Coxeter-like groups for groups of set-theoretic solutions of the Yang–Baxter equation

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- 1. Set-theoretic solutions of YBE, biracks and RC-quasigroups
- 2. YBE-groups and monoids
- 3. Garside germs and Coxeter-like groups

- Original Yang–Baxter Equation: For V a $\mathbb{C}$-vector space and $R : V \otimes V \to V \otimes V$,

$$R_{12}(a)\, R_{23}(a+b)\, R_{23}(b) = R_{23}(b)\, R_{23}(a+b)\, R_{12}(a). \qquad (*)$$

- Substituting R with PR, where $P(x \otimes y) := y \otimes x$, $(*)$ becomes

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}.$$

- Fact: If there exists a basis X of V s.t. R preserves $X \otimes X$ (a very special case!), then R is determined by the restriction of R to $X \otimes X$.

- Definition (Drinfel'd): A set-theoretic solution of YBE is a pair (X, ρ), where X is a set and $\rho : X \times X \to X \times X$ satisfies

$$\rho_{12}\, \rho_{23}\, \rho_{12} = \rho_{23}\, \rho_{12}\, \rho_{23}.$$

Called involutive if $\rho^2 = \mathrm{id}$, and nondegenerate if, writing $\rho = (\rho_1, \rho_2)$,

$$\forall s\ (y \mapsto \rho_1(s, y) \text{ is bijective}), \quad \text{and} \quad \forall t\ (x \mapsto \rho_2(x, t) \text{ is bijective}).$$

- Even for X finite, very poorly understood.

• <u>Definition</u> (Fenn–Rourke?): A birack is a triple $(X, \rceil, \lceil)$ where $\rceil, \lceil$ are binary operations on X satisfying

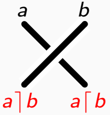
$$(a \rceil b) \rceil ((a \lceil b) \rceil c) = a \rceil (b \rceil c),$$
$$(a \rceil b) \lceil ((a \lceil b) \rceil c) = (a \lceil (b \rceil c)) \rceil (b \lceil c),$$
$$(a \lceil b) \lceil c = (a \lceil (b \rceil c)) \lceil (b \lceil c),$$

and the left-translations of $\rceil$ and the right-translations of $\lceil$ are one-to-one. A birack is involutive if, moreover,

$$(a \rceil b) \rceil (a \lceil b) = a \quad \text{and} \quad (a \rceil b) \lceil (a \lceil b) = b.$$

• <u>Proposition</u>: Invol. nondeg. set-theoretic solution YBE $\iff$ Involutive biracks.

• Proof: Put $a \rceil b := \rho_1(a, b)$, $a \lceil b := \rho_2(a, b)$, and use $(X, \rceil, \lceil)$ for colouring braids:



$\square$

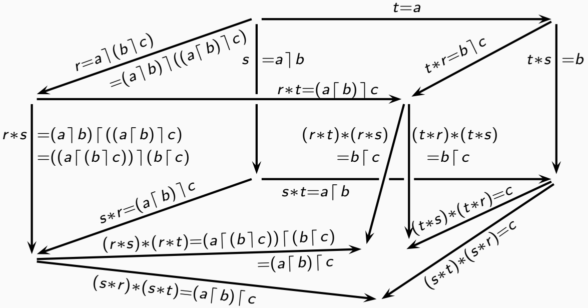
• Definition (Rump): An RC-system is a pair $(X, *)$ where $*$ is a binary operation on X satisfying

$$(x * y) * (x * z) = (y * x) * (y * z).$$

An RC-quasigroup is an RC-system whose left-translations are bijective.
An RC-system is bijective if $(s, t) \mapsto (s * t, t * s)$ is bijective.

• Proposition (Rump): Involutive biracks $\Longleftrightarrow$ Bijective RC-quasigroups.

• Proof: For $(X, \rceil, \lceil)$ an involutive birack, put $a * b :=$ unique c satisfying $a \rceil b = c$.
For $(X, *)$ a bijective RC-system, put $a \rceil b :=$ the unique c satisfying $a * b = c$.



- Definition (Etingof–Schedler–Soloviev): For (X, ρ) an invol. nondeg. set-theoretic solution of YBE, the structure group of (X, ρ) is

$$G := \langle X \mid \{ab = a'b' \mid (a', b') = \rho(a, b)\}\rangle.$$

- Equivalently: For $(X, *)$ a bijective RC-quasigroup, the structure group of $(X, *)$ is

$$G := \langle X \mid \{s(s * t) = t(t * s) \mid s, t \in X\}\rangle.$$

- Idem with monoids... $\langle ... \rangle^{+}$...

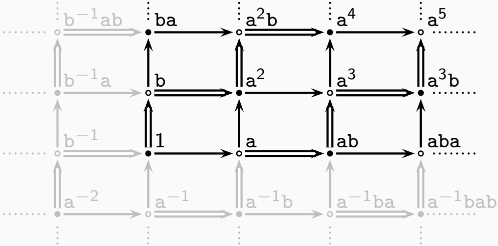
- Example: $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with $x * y = f(y)$ and $f : \mathtt{a} \mapsto \mathtt{b} \mapsto \mathtt{c} \mapsto \mathtt{a}$. Then

$$G := \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{ac} = \mathtt{b}^2, \mathtt{ba} = \mathtt{c}^2, \mathtt{cb} = \mathtt{a}^2\rangle.$$

- Main (open) question: Investigate "YBE-groups" and "YBE-monoids" from an algebraic and geometric viewpoint.

• Fact: The Cayley graph of an YBE-group (resp. monoid) with n atoms resembles that of $\mathbb{Z}^n$ (resp. $\mathbb{N}^n$).

• Example: $\langle \mathtt{a}, \mathtt{b} \mid \mathtt{a}^2 = \mathtt{b}^2 \rangle$ ($\simeq \mathbb{Z} \rtimes \mathbb{Z}$)



• Definition (Gateva–Van den Bergh): For M a monoid and $X \subseteq M$, an (X-based) I-structure for M is a bijection $\nu : \mathbb{N}^{(X)} \to M$ s.t. $\nu(1) = 1$, $\nu(s) = s$ for s in X, and

$$\forall a \in \mathbb{N}^{(X)}\ \exists \pi_a \in \mathfrak{S}_X\ \forall s \in X\ (\nu(as) = \nu(a)\, \pi_a(s)).$$

• Theorem (Gateva–Van den Bergh, Jespers–Okniński):

YBE-monoids with atom set X $\iff$ Monoids with an X-based I-structure.

- Theorem (Chouraqui):

YBE-groups with n atoms $\iff$ Garside groups with n atoms and $\binom{n}{2}$ quadratic relations s.t. every length-2 word appears in $\leqslant 1$ relation.

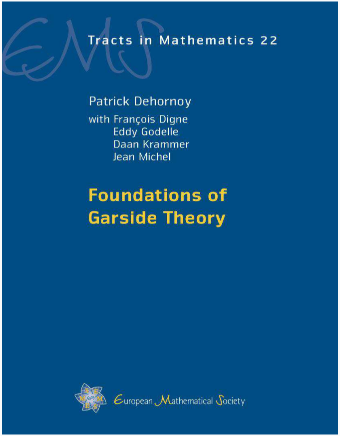
- Seminal example of a Garside group: Artin's braid group B_n.
  - B_n is a group of fractions of the monoid B_n^+,
  - every element of B_n^+ has a well-defined length in terms of the atoms σ_i,
  - any two elements of B_n^+ admit left- and right-lcms and gcds,
  - the left- and right-divisors of the half-turn braid Δ_n coincide, are finite in number, and generate B_n^+.

- Definition (D.–Paris): A Garside monoid is a pair (M, Δ) such that
  - M is a cancellative monoid and Δ belongs to M,
  - M is strongly noetherian (every element of M has a well-defined length),
  - any two elements of M admit left- and right-lcms and gcds,
  - the left- and right-divisors of Δ coincide, are finite in number, and generate M.

A Garside group is a group of fractions of a Garside monoid.

- Examples: B_n, but also: all spherical Artin–Tits groups (including $\mathbb{Z}^n$), many others, thus in particular YBE-groups.

- Main technical property of Garside groups: existence of a greedy normal form.
  - the latter extends to more general framework
    (no need of noetherianity assumption, no need of a "Garside element" Δ, etc.)
  - unifying notion of a Garside family (in a left-cancellative category)



- Example: (D.–Dyer–Hohlweg): Every Artin–Tits monoid admits a finite Garside family.

- In the case of braid groups: short exact sequence

$$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow \mathfrak{S}_n \longrightarrow 1,$$

with $\mathfrak{S}_n$ the size $n!$ quotient of B_n^+ obtained by collapsing σ_i^2 for every i,
+ a natural (set-theoretic) section $\sigma : f \mapsto \underline{f}$ from $\mathfrak{S}_n$ to B_n^+ s.t. $\underline{\mathfrak{S}}_n$ is a germ for B_n^+.

$$\langle\ \underline{\mathfrak{S}}_n \mid \{\underline{f}\,\underline{g} = \underline{h} \mid \ell(f) + \ell(g) = \ell(h)\}\ \rangle^+ = B_n^+$$

length of a permutation = number of inversions

- ▶ The whole structure of B_n^+ (and B_n) is encoded in the germ structure of the Coxeter group $\mathfrak{S}_n$ (formally: $\mathfrak{S}_n$ with the partial product $f \bullet g := fg$ if $\ell(fg) = \ell(f) + \ell(g)$)

- Question: Does there exist such a Coxeter-like quotient for every Garside group?

- Theorem (Bessis–Digne–Michel): YES for all spherical Artin–Tits groups.

the associated Coxeter group W is finite

$$1 \longrightarrow \pi_1(V^{\mathrm{reg}}) \longrightarrow G \longrightarrow W \longrightarrow 1.$$

- Definition: An RC-quasigroup $(X, *)$ is of class d if, for all s, t in X, one has

$$\Pi_{d+1}(s, ..., s, t) = t \quad (d \text{ times } s),$$

where $\Pi_1(x) = x$ and $\Pi_n(x_1, ..., x_n) = \Pi_{n-1}(x_1, ..., x_{n-1}) * \Pi_{n-1}(x_1, ..., x_{n-2}, x_n)$.

- Lemma (Rump): Every finite RC-quasigroup has a finite class.

- Theorem (D. 2015): "Coxeter-like quotients exist for all YBE-groups":
For G associated with an RC-system $(X, *)$ of size n and class d, there exists a short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow G \longrightarrow W \longrightarrow 1,$$

with W the size d^n quotient of G obtained by collapsing $s^{[d]}$ for every s in X,
plus a (set-theoretic) section $\sigma : f \mapsto \underline{f}$ from W to G^+ s.t. $\underline{W}$ is a germ for G^+.

$$s^{[d]} := \Pi_1(s)\Pi_2(s, s) \cdots \Pi_d(s, ..., s)$$

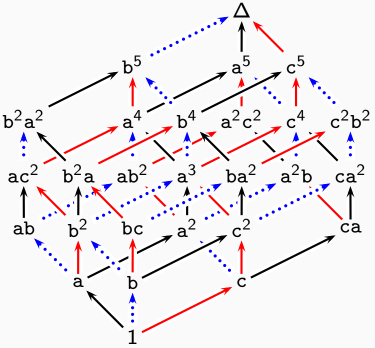
$$G^+ = \langle\, \underline{W} \mid \{\underline{f}\,\underline{g} = \underline{h} \mid \ell_X(f) + \ell_X(g) = \ell_X(h)\}\,\rangle^+$$

The whole structure of G^+ and G is encoded in the germ structure on W.

- Proof: Combines the I-structure and the Garside structure; key point: "RC-calculus".

- Remark: Special case of class 2 previously addressed by Chouraqui and Godelle.

- Example: Again $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with $x * y = f(y)$ and $f : \mathtt{a} \mapsto \mathtt{b} \mapsto \mathtt{c} \mapsto \mathtt{a}$.
  - Then $G := \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{ac} = \mathtt{b}^2, \mathtt{ba} = \mathtt{c}^2, \mathtt{cb} = \mathtt{a}^2 \rangle$.
  - The class is 3, leading to $W := \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{ac} = \mathtt{b}^2, \mathtt{ba} = \mathtt{c}^2, \mathtt{cb} = \mathtt{a}^2, \mathtt{abc} = 1 \rangle$.



- Question: Which finite groups arise? What are their linear representations?

(known: for $\#X = n$, there exists an n-dimensional unitary representation)

e.g., above: $\mathtt{a} \mapsto \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathtt{b} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}$, $\mathtt{c} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix}$

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