Garside and quadratic normalisation: a survey

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Vacations in North Wales, July 2015

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Arlette at the top of Craig Cow

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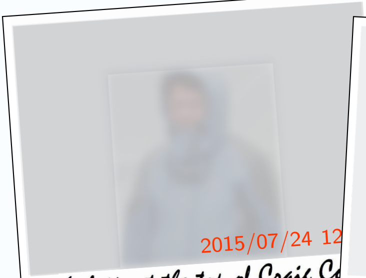

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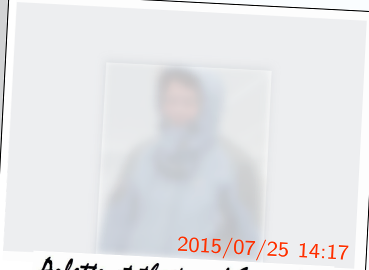

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- A survey of normal forms in monoids that are
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- A common mechanism inducing a universal recipe: the domino rule.

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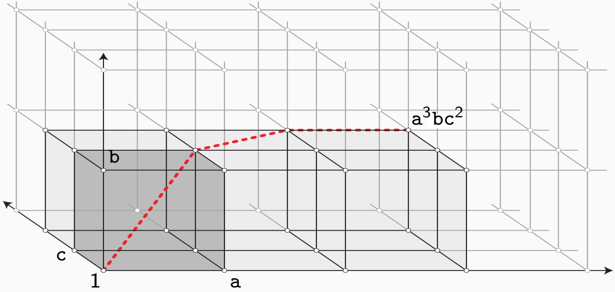

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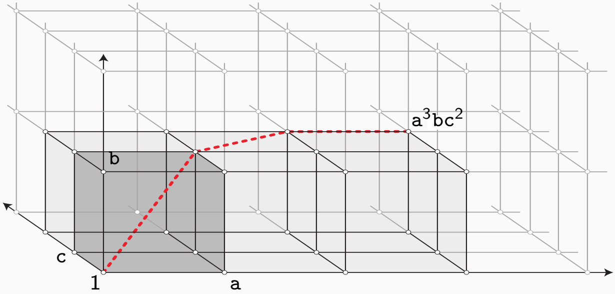

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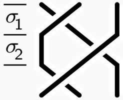

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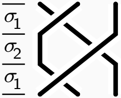

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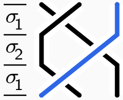

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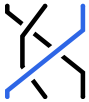

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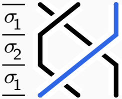

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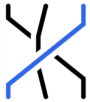

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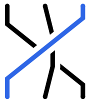

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continuous deformation of the ambient 3D-space

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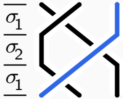

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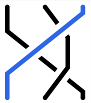

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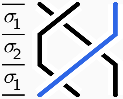

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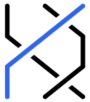

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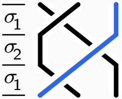

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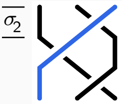

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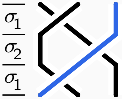

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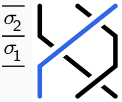

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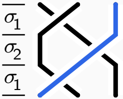

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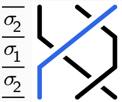

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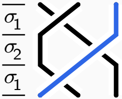

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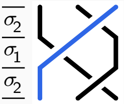

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any two strands cross at most once
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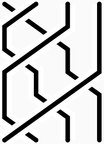

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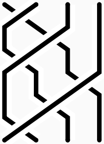

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▶ the greedy (or Garside) normal form $\mathrm{NF}^{\mathrm{Gar}}(g)$ (with respect to S_n).

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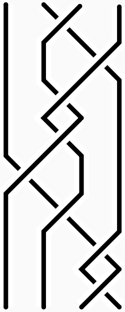

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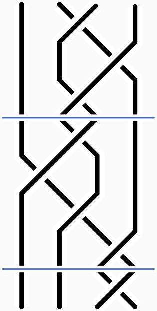

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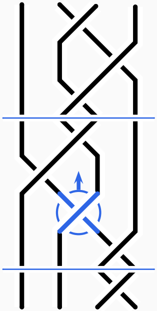

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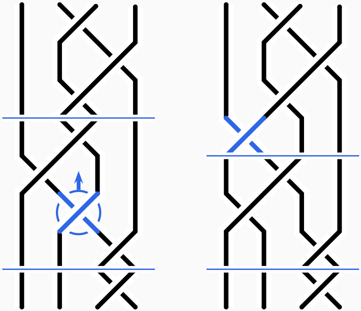

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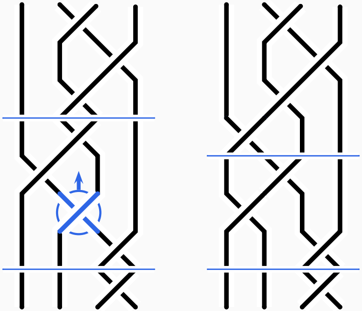

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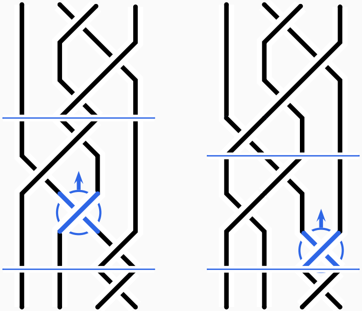

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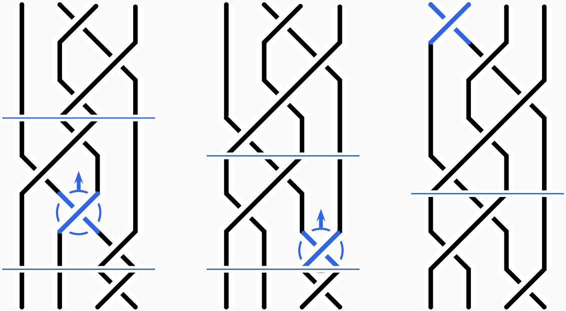

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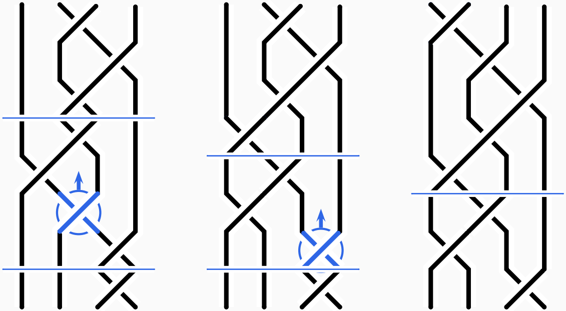

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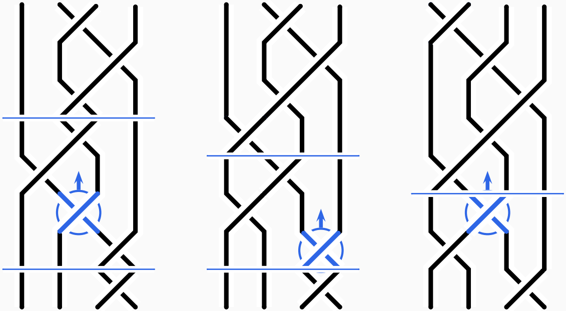

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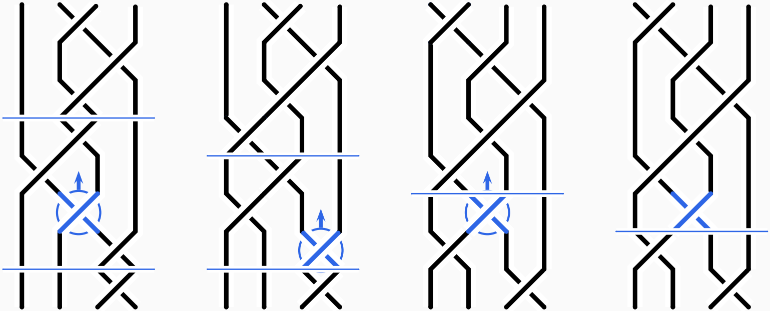

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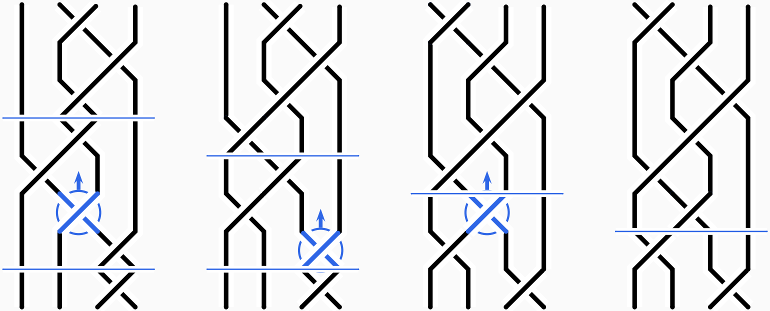

Plan:

- 1. Two examples
  - Free abelian monoids
  - Braid monoids
- 2. Garside normalisation
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  - Artin–Tits monoids
- 3. Quadratic normalisation
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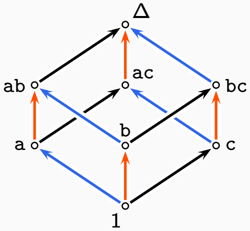

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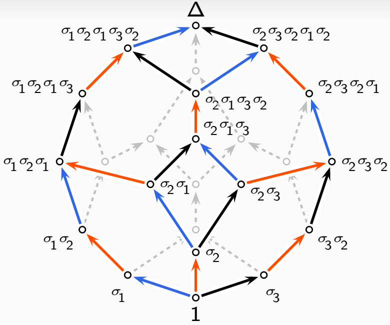

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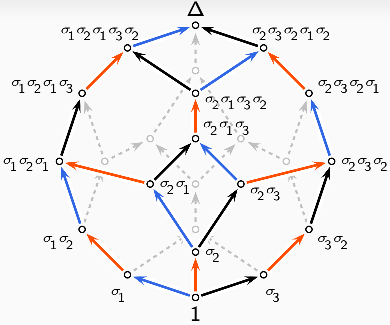

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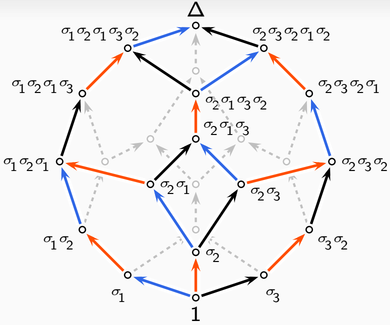

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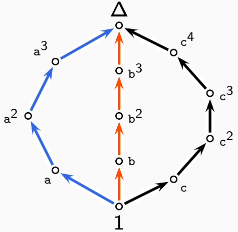

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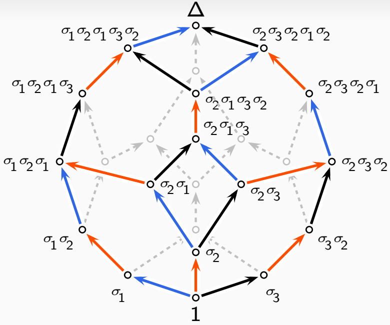

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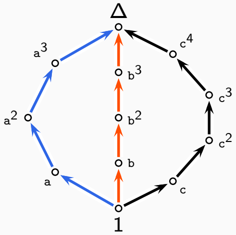

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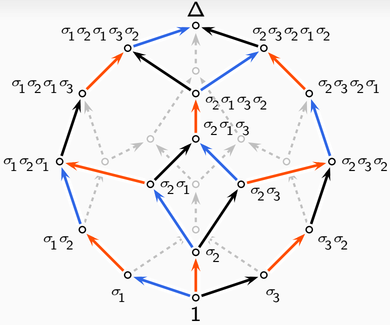

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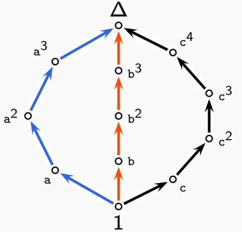

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- Go to a more general scheme: Garside families.

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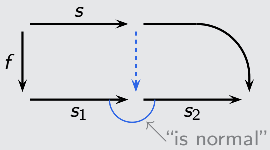

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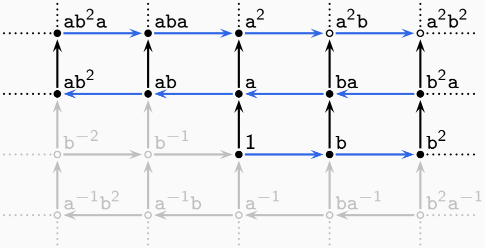

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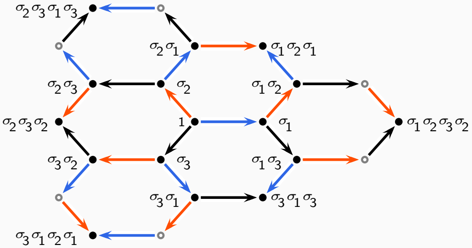

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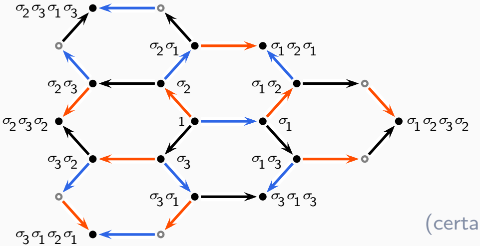

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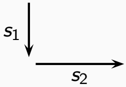

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$$\begin{tikzcd}
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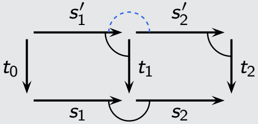

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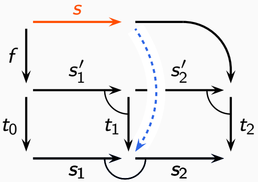

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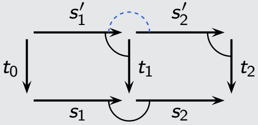

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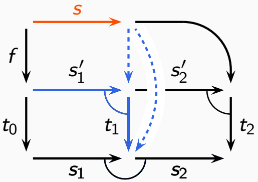

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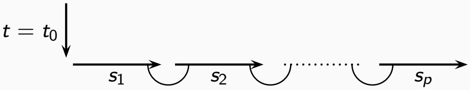

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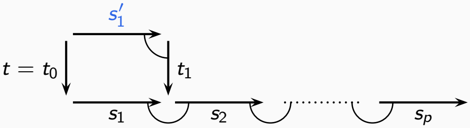

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{} \arrow[d, "s_1"'] \\
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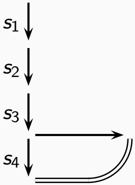

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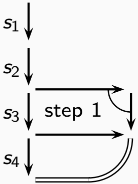

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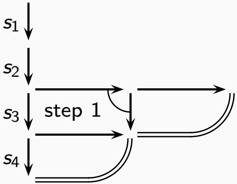

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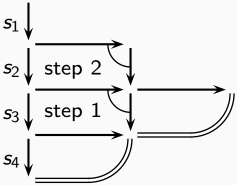

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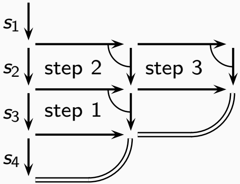

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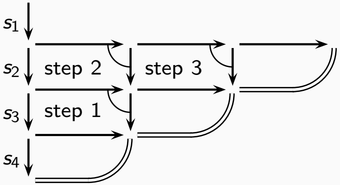

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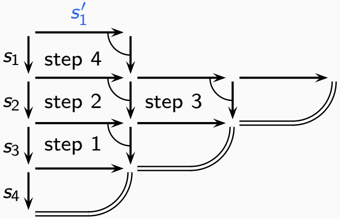

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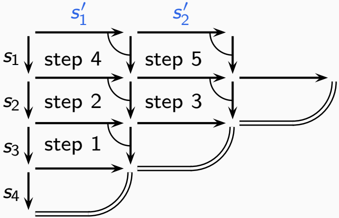

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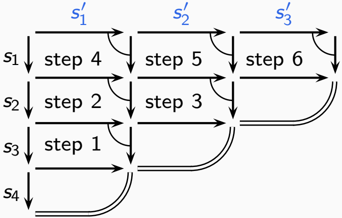

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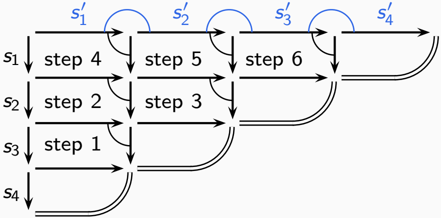

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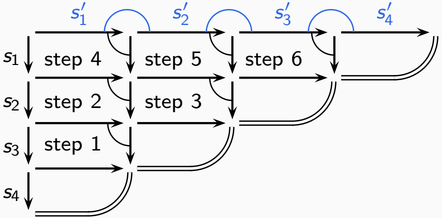

- Corollary: *If a monoid M is left-cancellative, has no invertible element $\neq 1$, and admits a finite Garside family S:*
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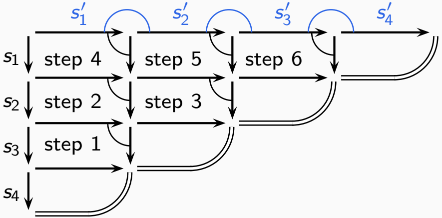

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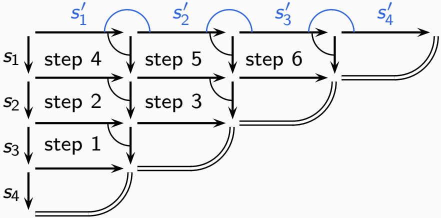

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  - (Picantin) *M is an automaton semigroup and is residually finite.*

Plan:

- 1. Two examples
  - Free abelian monoids
  - Braid monoids

- 2. Garside normalisation
  - Garside monoids
  - Artin–Tits monoids

- 3. Quadratic normalisation
  - Plactic monoids

- From now on: consider (more) general geodesic normal forms for a monoid.

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the normal form has minimal length

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Id. for every Garside family S in a left-cancellative monoid M.

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    a sequence of positions u (depending on w) s.t. $N(w) = \overline{N}_u(w)$.

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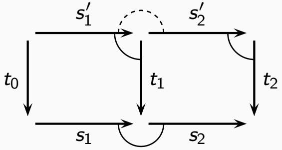

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(recall: $\delta_2 = 1$, $\delta_3 = 212$, $\delta_4 = 323123$, $\delta_5 = 4342341234$, etc.)

- Lemma: *A quadratic normalisation (S, N) is of class $(4, 3)$ iff the domino rule is valid for (S, N).*

$$\begin{tikzcd} \bullet \arrow[r, "s_1'"] \arrow[d, "t_0"'] & \bullet \arrow[r, "s_2'"] \arrow[d, "t_1"] & \bullet \arrow[d, "t_2"] \\ \bullet \arrow[r, "s_1"'] & \bullet \arrow[r, "s_2"'] & \bullet \end{tikzcd}$$

  - Hence: The mechanism for class (4,3) is the same as in the Garside case.

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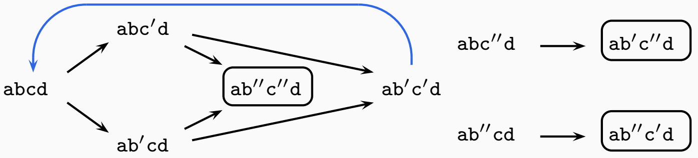

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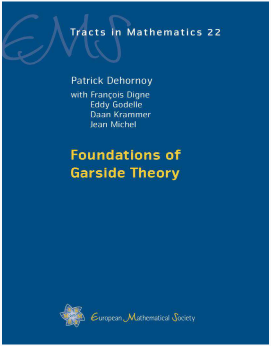

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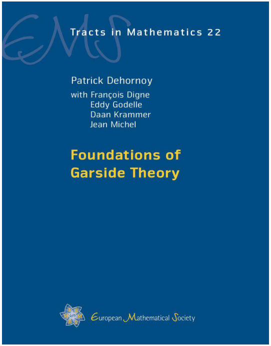

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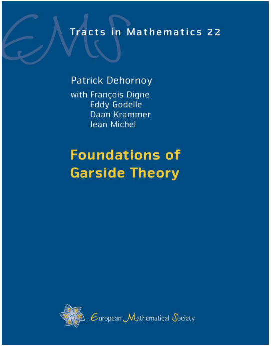

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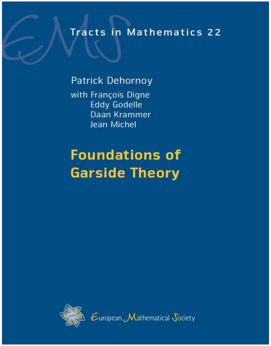

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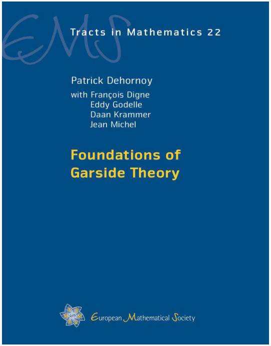

Part 3:

• P. Dehornoy, Y. Guiraud, *Quadratic normalisation in monoids* arXiv:1504.02717

• A. Hess, V. Ozornova, *Factorability, string rewriting and discrete Morse theory* arXiv:1412.3025