Garside and quadratic normalisation: a survey

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- A survey of normal forms in monoids that are
  - based on greedy algorithms (Garside normalisation),
  - and, more generally, on local algorithms (quadratic normalisation).
- A common mechanism inducing a universal recipe: the domino rule.

Plan:

- 1. Two examples
  - Free abelian monoids
  - Braid monoids
- 2. Garside normalisation
  - Garside monoids
  - Artin–Tits monoids
- 3. Quadratic normalisation
  - Plactic monoids

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- Let M be a free abelian monoid based on $A_n := \{a_1, ..., a_n\}$ ($\simeq (\mathbb{N}, +)^n$).
  - each element of M has an A_n-decomposition that is unique up to the order of letters;

- Fix a linear order $\leqslant$ on A_n.
  - each element of M has a unique A_n-decomposition $s_1 | \cdots | s_p$ with $s_1 \leqslant \cdots \leqslant s_p$:
  - the lexicographic normal form $\mathrm{NF}^{\mathrm{Lex}}(g)$ (with respect to $\leqslant$).

- Another (more complicated, but more easily extendible) normal form:
  - put $S_n := \{\prod_{i \in I} a_i \mid I \subseteq \{1, ..., n\}\}$ (so $\# S_n = 2^n$)

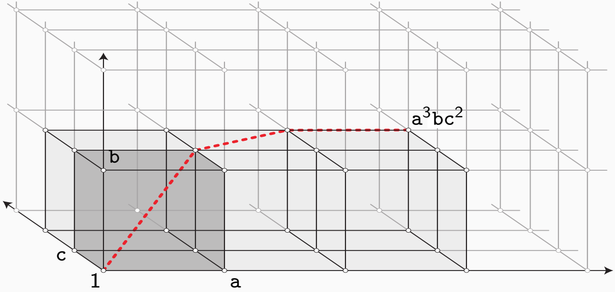
- Proposition: *Each element of M has a unique S_n-decomposition $s_1 | \cdots | s_p$ with $s_p \neq 1$, and*
$$\forall s \in S_n \, (s_i \prec s \Rightarrow s \not\preccurlyeq s_i s_{i+1} \cdots s_p). \qquad (*)$$

s_i is a proper divisor of s: $\exists t \neq 1 \, (s_i t = s)$ $\neg \exists t \, (st = s_i s_{i+1} \cdots s_p)$

hence: $(*)$ means: "s_i is a maximal (left)-divisor of $s_i s_{i+1} \cdots s_p$ lying in S_n"

  - the greedy normal form $\mathrm{NF}^{\mathrm{Gar}}(g)$ (with respect to S_n).

- Example: $\mathrm{NF}^{\mathrm{Gar}}(\mathtt{a}^3\mathtt{b}\mathtt{c}^2) = \mathtt{abc}|\mathtt{ac}|\mathtt{a}$.



- Definition: The n-strand braid monoid is

$$B_n^+ := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle^+ .$$

- Theorem (Artin 1926, Garside 1969): Under the correspondence

$\sigma_i \quad \leftrightsquigarrow$ 

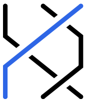
and concatenation (stacking) of diagrams, the elements of B_n^+ interpret as isotopy classes of positive n-strand braid diagrams.

↑
continuous deformation of the ambient 3D-space

- Topological interpretation of the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$:

σ_1
σ_2
σ_1

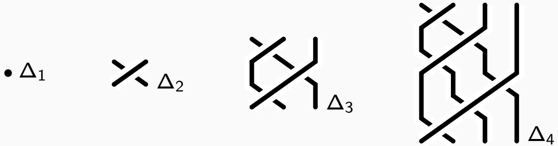

is isotopic to

σ_2
σ_1
σ_2


any two strands cross at most once

• Put S_n := {simple n-strand braids} $= \{g \in B_n^+ \mid g \preccurlyeq \Delta_n\}$.

$\exists h\ (gh = \Delta_n)$ the half-turn braid:



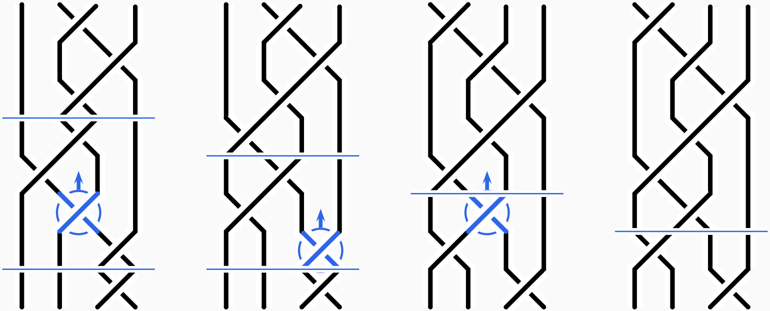
• Proposition (Adyan 1984, Morton–El-Rifai 1988): *Every element g of B_n^+ has a unique decomposition $s_1 | \cdots | s_p$ with $s_1, ..., s_p \in S_n$, $s_p \neq 1$, and*

$$\forall s \in S_n\, (s_i \prec s \Rightarrow s \not\preccurlyeq s_i s_{i+1} \cdots s_p).$$

i.e., again: "s_i is a maximal left-divisor of $s_i s_{i+1} \cdots s_p$ lying in S_n"

▸ the greedy (or Garside) normal form $\mathrm{NF}^{\mathrm{Gar}}(g)$ (with respect to S_n).

- <u>Example</u>: $\mathrm{NF}^{\mathrm{Gar}}(\sigma_2\sigma_3\sigma_2^2\sigma_1\sigma_2\sigma_3^2) = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2|\sigma_1\sigma_3$.



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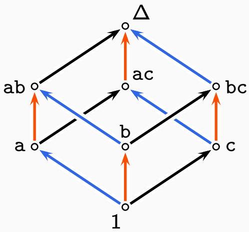
- Definition: A Garside monoid is a pair (M, Δ), where M is a cancellative monoid s.t.
  - There exists $\lambda : M \to \mathbb{N}$ satisfying, for all f, g,
  $$\lambda(fg) \geqslant \lambda(f) + \lambda(g) \qquad \text{and} \qquad g \neq 1 \Rightarrow \lambda(g) \neq 0.$$
  - Any two elements of M admit left- and right-lcms and gcds.
  - Δ is a Garside element of M, meaning: the left- and the right-divisors of Δ coincide and generate M.
  - The family $\mathrm{Div}(\Delta)$ of all divisors of Δ in M is finite.

- Philosophy: The finite lattice $\mathrm{Div}(\Delta)$ encodes the whole structure of M.

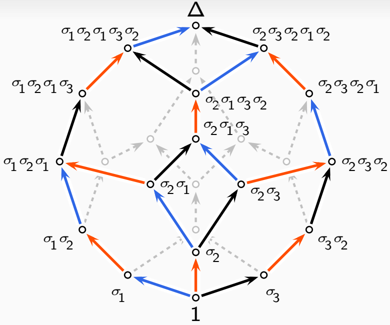
- Example: Put $\Delta_n := \mathtt{a}_1 + \cdots + \mathtt{a}_n$.

Then $(\mathbb{N}^n, \Delta_n)$ is a Garside monoid.

  - Here the lattice $\mathrm{Div}(\Delta_n)$ is an n-dimensional cube (2^n elements):



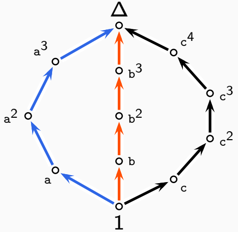
- <u>Example</u>: (B_n^+, Δ_n) is a Garside monoid.
  - Here the lattice $\mathrm{Div}(\Delta_n)$ is the nth <u>permutohedron</u> ($n!$ elements):



- <u>Example</u>: $M = \langle \mathtt{a}_1, \dots, \mathtt{a}_n \mid \mathtt{a}_1^{e_1} = \cdots = \mathtt{a}_n^{e_n} \rangle^+$ with $\Delta = \mathtt{a}_1^{e_1} = \cdots = \mathtt{a}_n^{e_n}$.

Then (M, Δ) is a Garside monoid.

  - Here the lattice $\mathrm{Div}(\Delta)$ has $e_1 + \cdots + e_n - n + 2$ elements:



and many more... ask Matthieu Picantin!

• Proposition: *If (M, Δ) is a Garside monoid, every element g of M has a unique decomposition $s_1|\cdots|s_p$ satisfying $s_1, \ldots, s_p \in Div(\Delta)$, $s_p \neq 1$, and*

$$\forall s{\in}Div(\Delta)\,(s_i \prec s \Rightarrow s \not\preccurlyeq s_i s_{i+1} \cdots s_p).$$

once more: s_i is a maximal left-divisor of $s_i s_{i+1} \cdots s_p$ lying in $\mathrm{Div}(\Delta)$.

- A "greedy" normal form

• Proof (existence): Left-dividing s and Δ means left-dividing $\gcd_L(s, \Delta)$.

- Write $g = s_1 g'$ with $s_1 = \gcd_L(g, \Delta)$.
- Then iterate: $g' = s_2 g''$, $g'' = s_3 g'''$, etc. □

• Question: How to effectively compute this normal form? What is the mechanism?

- Go to a more general scheme: Garside families.

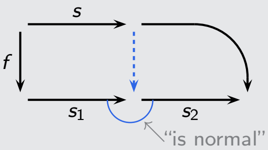
- <u>Convention</u>: associate with an element g of a monoid an arrow $\xrightarrow{\quad g \quad}$;
  - then $\xrightarrow{\quad f \quad}\ \xrightarrow{\quad g \quad}$ for fg, ($=$ think of the monoid as of a category)
  - and
$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "f"'] & \cdot \arrow[d, "g'"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$
for $fg = f'g'$.

- <u>Definition</u>: (i) If M is a left-cancellative monoid and $S \subseteq M$, call an S-word $s_1|s_2$ S-normal if

$$\forall s{\in}S\ \forall f{\in}M\ (s \preccurlyeq fs_1s_2 \Rightarrow s \preccurlyeq fs_1),$$

and $s_1|\cdots|s_p$ S-normal iff $s_i|s_{i+1}$ is S-normal for each i.

(ii) Call S a Garside family if every element of M admits an S-normal decomposition.

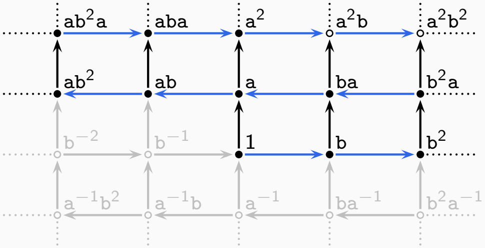


- <u>Lemma</u>: *If (M, Δ) is a Garside monoid, then $Div(\Delta)$ is a Garside family in M; an S-word is S-normal for $S := Div(\Delta)$ <u>iff</u> it is normal in the sense of Garside monoids.*
  - Hence: we recover the previous framework...

... but also catch new examples:

- Example (stupid): Every left-cancellative monoid is a Garside family in itself.
  - Only proper (finite) subfamilies may be interesting.

- Example: ("Klein bottle monoid") Let $K^+ := \langle \mathtt{a}, \mathtt{b} \mid \mathtt{a} = \mathtt{bab} \rangle^+$.
  Then $\mathrm{Div}(\mathtt{a}^2)$ is a Garside family in M.



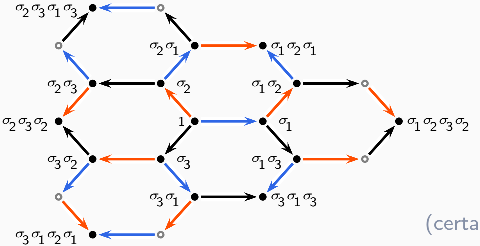
- Theorem: *Assume that M is a left-cancellative monoid that is noetherian and any two elements of M admit a unique left-gcd. Then S is a Garside family in M iff S*
  - *contains the atoms of M,*
  - *is closed under right-lcm (if two elements of S have a right-lcm, it lies in S),*
  - *and is closed under right-divisor.*

  - In this case, there must exist a smallest Garside family.

- Example (D.–Dyer–Hohlweg): Every finitely generated Artin–Tits monoid admits a finite Garside family.

defined by relations $sts... = tst...$, same length

Typically ("type $\widetilde{A}_2$"): $\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_3\sigma_1 \rangle^+$



(certainly not a lattice)

- The Garside normal form NF^S is indeed a greedy normal form:

- Proposition: *If S is a Garside family in a left-cancellative monoid M:*
  - *The S-normal form is (essentially) unique when it exists.*
  - *If S is finite, the language of S-normal words is regular.*
  - *A word $s_1|\cdots|s_p$ is S-normal iff*

$$\forall s{\in}S\,(s_i \prec s \Rightarrow s \not\preccurlyeq s_i s_{i+1} \cdots s_p).$$

once again: s_i is a maximal left-divisor of $s_i s_{i+1} \cdots s_p$ lying in S

- Main question: How to compute the S-normal form? What is the mechanism?
  - Equivalently: how to compute the normalisation map $N^S : S^* \to S^*$?

- Lemma: *If S is a Garside family in a left-cancellative monoid M, then, for all s_1, s_2 in S, the S-normal form of $s_1 s_2$ has length $\leqslant 2$.*
  - Makes sense to consider the restriction $\overline{N}^S := N^S \restriction S^{[2]}$

$$\begin{tikzcd} {} \arrow[r, dashed, "t_1"] \arrow[d, "s_1"'] & {} \arrow[d, dashed, "t_2"] \\ {} \arrow[r, "s_2"'] & {} \end{tikzcd}$$

- <u>Lemma</u> ("domino rule"): *If S is a Garside family in a left-cancellative monoid M,*

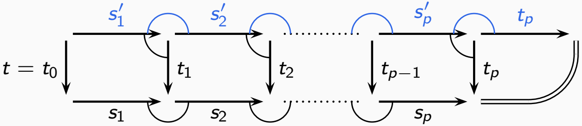
$$\begin{tikzcd} \cdot \arrow[r, "s'_1"] \arrow[d, "t_0"'] & \cdot \arrow[r, "s'_2"] \arrow[d, "t_1"] & \cdot \arrow[d, "t_2"] \\ \cdot \arrow[r, "s_1"'] & \cdot \arrow[r, "s_2"'] & \cdot \end{tikzcd}$$

  - <u>Proof</u>:

$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "f"'] & \cdot \arrow[d, dashed] \arrow[dd, dashed, bend left] \arrow[dr, bend left] & \\ \cdot \arrow[r, "s'_1"] \arrow[d, "t_0"'] & \cdot \arrow[r, "s'_2"] \arrow[d, "t_1"'] & \cdot \arrow[d, "t_2"] \\ \cdot \arrow[r, "s_1"'] & \cdot \arrow[r, "s_2"'] & \cdot \end{tikzcd}$$

$\square$

• Proposition: *If S is a Garside family in a left-cancellative monoid M, and $s_1|\cdots|s_p$ is S-normal, and t lies in S, then the S-normal form of $ts_1\cdots s_p$ is*



,

that is, $N^S(t|s_1|\cdots|s_p) = \overline{N}^S_{1|2|\cdots|p-1}(t|s_1|\cdots|s_p)$.

$\uparrow$ applying $\overline{N}^S := N^S\restriction S^{[2]}$ in positions 1, then 2, etc. until $p-1$

• Corollary: *If S is a Garside family in a left-cancellative monoid M:*

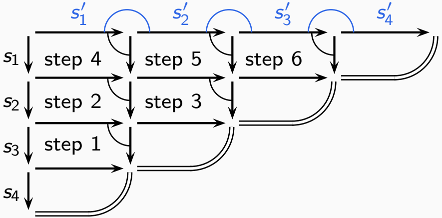
- *For each t in S, there is a rational transducer computing $N(tw)$ from $N(w)$.*
- *Garside normalisation satisfies the 2-Fellow Traveller Property on the left.*

- Iterating from the right: a <u>universal</u> recipe for normalising words of length p:

- <u>Theorem</u>: *If S is a Garside family in a left-cancellative monoid M, and w lies in $S^{[p]}$, the S-normal form of w is given by*

$$N^S(w) = \overline{N}^S_{\delta_p}(w),$$

with $\delta_2 := 1$, $\delta_3 := 2|1|2$, $\delta_4 := 3|2|3|1|2|3$, $\delta_5 := 4|3|4|2|3|4|1|2|3|4$, *etc.*



- <u>Corollary</u>: *If a monoid M is left-cancellative, has no invertible element $\neq 1$, and admits a <u>finite</u> Garside family S:*
  - *N^S can be computed in* $\mathrm{DTIME}(n^2)$, *and the Word Pb for (M, S) lies in* $\mathrm{DTIME}(n^2)$.
  - *If M is right-cancellative, M is left-automatic.*
  - (Picantin) *M is an automaton semigroup and is residually finite.*

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• From now on: consider (more) general geodesic normal forms for a monoid.

the normal form has minimal length

• Proposition: *There exists a notion of a normalisation (S, N), with N a length preserving map $S^* \to S^*$, s.t. defining a geodesic normal form on a monoid M is equivalent to defining a normalisation mod a neutral letter for M.*

a letter e satisfying $\forall w\ (N(w|e) = N(e|w) = N(w)|e)$
$M = \langle S \mid \{w = N(w) \mid w \in S^*\} \cup \{e = 1\}\rangle^+$

• Example (lexicographic): $M = \mathbb{N}^n$ and $N^{\mathrm{Lex}}(w) := w$ lexicographically sorted.

• Example (Garside): $M = B_n^+$, $S = \mathrm{Div}(\Delta_n)$, and $N^{\mathrm{Gar}}(s_1|\cdots|s_p) := (s_1'|\cdots|s_q'|1|\cdots|1)$, with $s_1'|\cdots|s_q'$ the S-normal form of $s_1 \cdots s_p$.

Id. for every Garside family S in a left-cancellative monoid M.

- Definition: A normalisation (S, N) is quadratic if
  - An S-word w is N-normal $(=$ fixed under $N)$ iff every length-2 factor of w is,
  - One can go from w to $N(w)$ by normalising length-2 factors.

(independent conditions: neither implies the other)

- Examples:
  - (S, N^{Lex}) is quadratic: a word is $<^{\mathrm{Lex}}$-nondecreasing iff every length-2 factor is, and one can from w to $N^{\mathrm{Lex}}(w)$ by swapping adjacent letters.
  - (S, N^{Gar}) is quadratic: a word is S-normal iff every length-2 factor is, and one can from w to $N^{\mathrm{Gar}}(w)$ by normalising length-2 factors: domino rule.

- Fact: *If (S, N) is a quadratic normalisation, the set of N-normal words is regular.*

- Notation: For (S, N) quadratic: $\overline{N} := N{\restriction}S^{[2]}$,

  $\overline{N}_i := \overline{N}$ applied to the factor in position $i, i+1$,

  $\overline{N}_{i_1|\cdots|i_m} := \overline{N}_{i_m} \circ \cdots \circ \overline{N}_{i_1}$,

  - If (S, N) is quadratic, there exists for every S-word w a sequence of positions u (depending on w) s.t. $N(w) = \overline{N}_u(w)$.

- For $\|w\| = 3$, the only possibilities are $u = 121...[c]$ or $u = 212...[c]$.

$$\underset{1|2|1|\ldots,\ \text{length } c}{\underset{\uparrow}{121...[c]}}$$

- Definition: A quadratic normalisation (S, N) is of left class c if

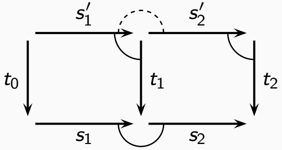
$$\forall w{\in}S^{[3]}\ (N(w) = \overline{N}_{121...[c]}(w)).$$

... right class c ... $\overline{N}_{212...[c]}(w))$ ...
... class (c, c') for left class c and right class c'.

- Lemma: *If (S, N) is of left class c, then*
  - *(S, N) is of left class c' for every $c' \geqslant c$,*
  - *(S, N) is of right class c'' for every $c'' \geqslant c{+}1$.*

- Examples:
  - (S, N^{Lex}) is of class $(3, 3)$: $\forall w{\in}S^{[3]}\ (N^{\mathrm{Lex}}(w) = \overline{N}_{121}(w) = \overline{N}_{212}(w))$.
  - (S, N^{Gar}) is of class $(4, 3)$: $\forall w{\in}S^{[3]}\ (N^{\mathrm{Gar}}(w) = \overline{N}_{1212}(w) = \overline{N}_{212}(w))$.
  - Define $N^{\mathrm{Lex}}_{*}(s|t) := \lceil (s+t)/2 \rceil | \lfloor (s+t)/2 \rfloor$ for $s > t$, and $s|t$ otherwise.
  Then $(S, N^{\mathrm{Lex}}_{*})$ is of (minimal) class $(3{+}\lfloor \log_2(n) \rfloor, 3{+}\lfloor \log_2(n) \rfloor)$, where $n = \#S$.

- <u>Lemma</u>: *A quadratic normalisation (S,N) is of class $(4,3)$ <u>iff</u> the domino rule is valid for (S,N).*



- ▶ Hence: The mechanism for class (4,3) is the same as in the Garside case.

- <u>Proposition</u>: *If (S,N) is of class $(4,3)$, then, for every length-p word w, one has*

$$N(w) = \overline{N}_{\delta_p}(w).$$

(recall: $\delta_2 = 1$, $\delta_3 = 212$, $\delta_4 = 323123$, $\delta_5 = 4342341234$, etc.)

- Catch new examples with the same mechanism:

- Definition: For $(X, <)$ a totally ordered set, the plactic monoid on $(X, <)$ is

$$P_X := \left\langle\ X \ \middle|\ \begin{array}{ll} \mathtt{acb} = \mathtt{cab} & \text{for } \mathtt{a} \leqslant \mathtt{b} < \mathtt{c} \\ \mathtt{bac} = \mathtt{bca} & \text{for } \mathtt{a} < \mathtt{b} \leqslant \mathtt{c} \end{array} \right\rangle^{+}.$$

- Connection with Young tableaux:
  - Another family of generators: $S := \{\text{columns over } X\}$
    $:=$ strictly decreasing products of elements of X.
  - Call a pair of columns $c|c'$ normal for
    $$\|c\| \geqslant \|c'\| \quad \& \quad \forall k \leqslant \|c'\| \ (c_k \leqslant c'_k).$$
  - A geodesic normal form on (P_X, S), computed by Schensted's insertion algorithm.

- Proposition: *Schensted normalisation is quadratic of class* $(3,3)$.

- Similar for the Chinese monoids, now with class $(5,5)$.

• Theorem (axiomatisability): *If* (S, N) *is of class* $(4,3)$, *then* $\overline{N}$ $(= N{\restriction}S^2)$ *satisfies*

$$\overline{N}_{212} = \overline{N}_{1212} = \overline{N}_{2121}.$$

Conversely, if $F : S^{[2]} \to S^{[2]}$ *satisfies*

$$F_{212} = F_{1212} = F_{2121},$$

then there exists a unique normalisation (S, N) *of class* $(4,3)$ *satisfying* $\overline{N} = F$.

• Definition: Call a (quadratic) normalisation (S, N) left-weighted if

$\forall s, t, s', t'$ $(s'|t' = N^{\mathrm{Gar}}(s|t) \Longrightarrow s$ left-divides s' in the associated monoid).

• Theorem (characterization): *If* M *is a left-cancellative monoid and* S *is a Garside family in* M, *then* (S, N^{Gar}) *is of class* $(4,3)$ *and is left-weighted.*

Conversely, if (S, N) *is a left-weighted class* $(4,3)$ *normalisation, then* S *is a Garside family in* M *and* $N = N^{Gar}$ *holds.*

- With each normalisation (S, N) comes a rewrite system:

$$\text{rules: } s|t \to \overline{N}(s|t) \quad \text{when } \overline{N}(s|t) \neq s|t.$$

  - then normalising: $\forall w\ \exists w'$normal $(w \to^* w')$,
  - and confluent: $\forall w, w', w''\ ((w \to^* w' \,\&\, w \to^* w'') \Rightarrow \exists w'''\ (w' \to^* w''' \,\&\, w'' \to^* w'''))$.
  - but is it terminating: is <u>every</u> rewriting sequence finite?

- <u>Proposition</u>: *There exists a nonterminating class* $(4, 4)$ *normalisation.*

  - <u>Proof</u>: $\mathtt{ab} \to \mathtt{ab}'$, $\mathtt{cd} \to \mathtt{c}'\mathtt{d}$, $\mathtt{bc}' \to \mathtt{b}''\mathtt{c}''$, $\mathtt{b}'\mathtt{c} \to \mathtt{b}''\mathtt{c}''$, $\mathtt{b}'\mathtt{c}' \to \mathtt{bc}$.

$$\begin{tikzcd}
 & \mathtt{abc'd} \arrow[dr] \arrow[drr] & & & \mathtt{abc''d} \arrow[r] & \boxed{\mathtt{ab'c''d}} \\
\mathtt{abcd} \arrow[ur] \arrow[dr] & & \boxed{\mathtt{ab''c''d}} & \mathtt{ab'c'd} \arrow[lll, bend right] & & \\
 & \mathtt{ab'cd} \arrow[ur] \arrow[urr] & & & \mathtt{ab''cd} \arrow[r] & \boxed{\mathtt{ab''c'd}}
\end{tikzcd}$$

□

• Proposition: *Every class (3,3) normalisation is terminating: every rewriting sequence from a length-p word has length at most $p(p-1)/2$.*

▶ Proof: Uses Matsumoto's lemma for the symmetric group. □

• Theorem: *Every class (4,3) normalisation is terminating: every rewriting sequence from a length-p word has length at most $2^p - p - 1$.*

▶ Proof: Because of the domino rule, one inevitably proceeds to the normal form. □

• Corollary: *Every Garside normalisation is terminating.*

• Application: *Every finite type Artin–Tits monoid has a finite converging presentation.*

▶ Proof: Take for S a finite Garside family, with relations $s|t = N^{\mathrm{Gar}}(s|t)$. □

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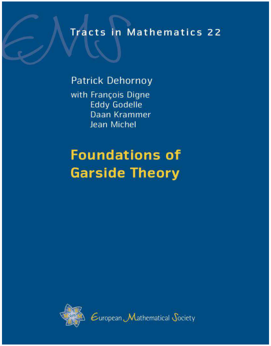

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