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and the Word Problem for Artin-Tits groups

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  - based on a rewrite system extending free reduction,
  - reminiscent of the Dehn algorithm for hyperbolic groups,
  - proved in particular cases, conjectured in the general case.

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  - Free reduction
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  - consider more general rewrite rules.

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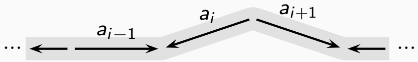

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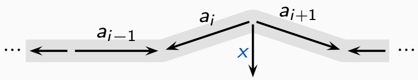

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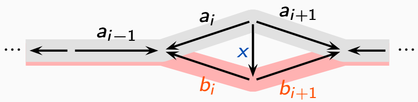

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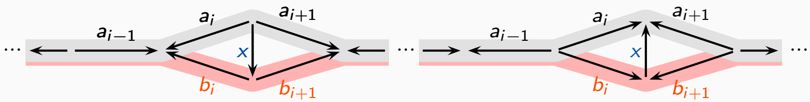

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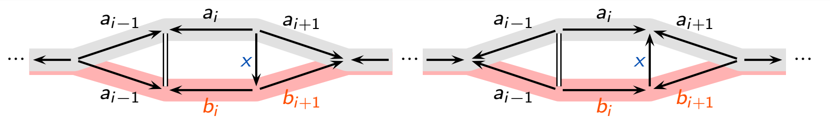

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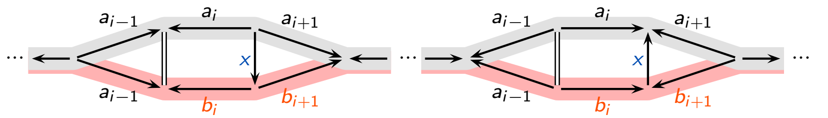

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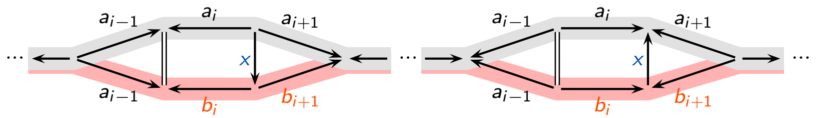

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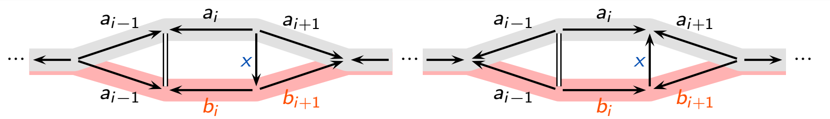

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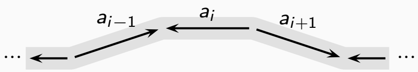

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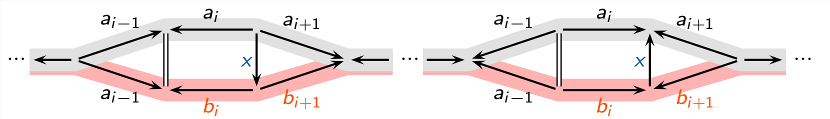

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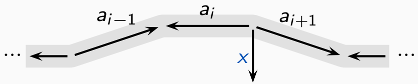

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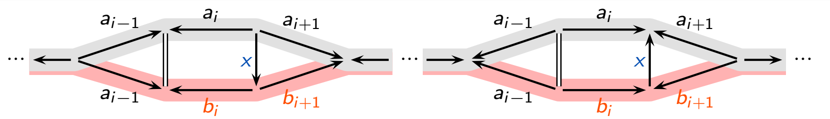

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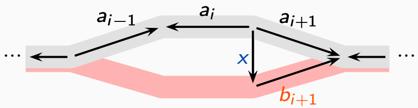

- Diagrammatic representation of elements of M: $\xrightarrow{a} \ \mapsto a$, and multifractions:

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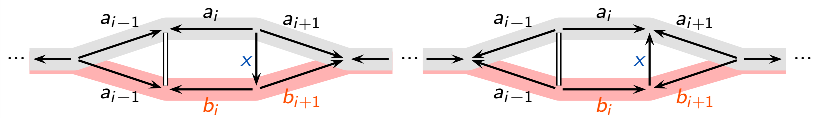

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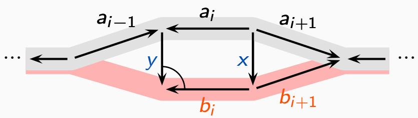

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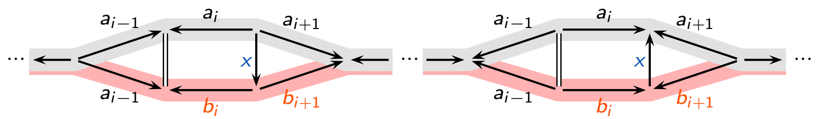

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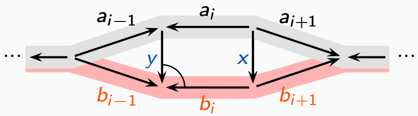

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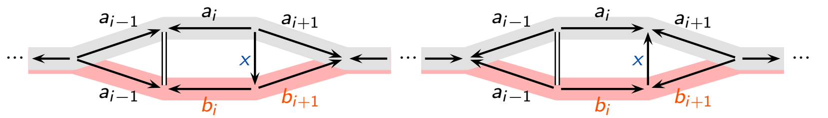

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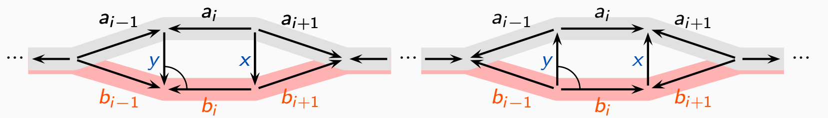

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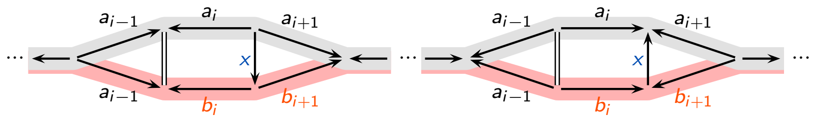

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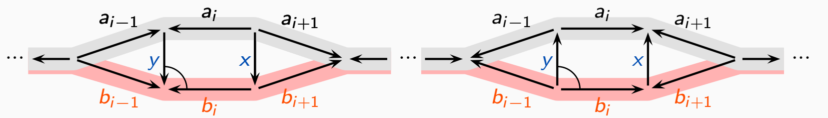

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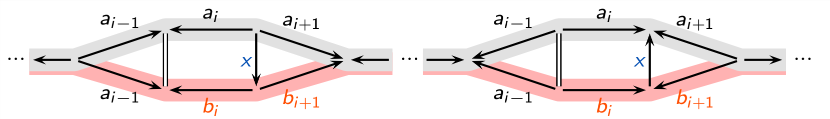

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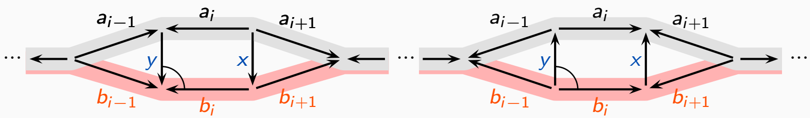

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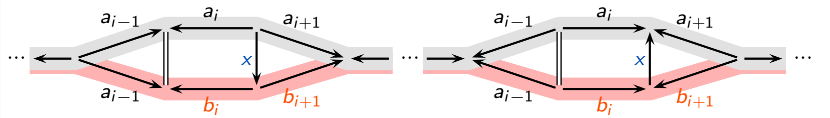

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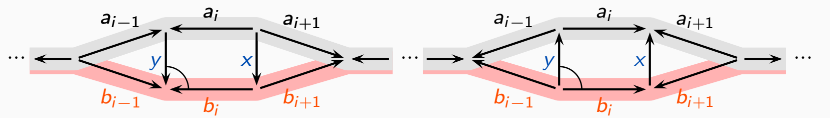

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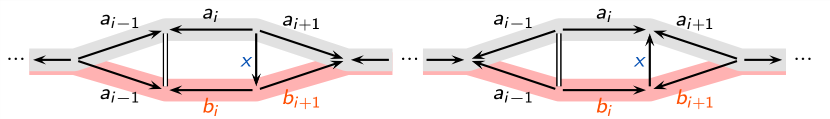

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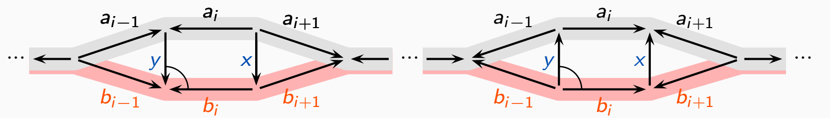

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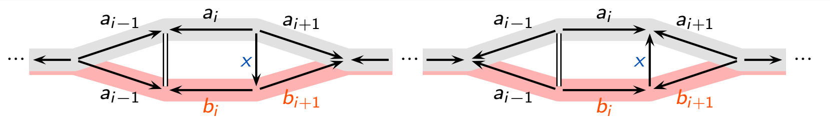

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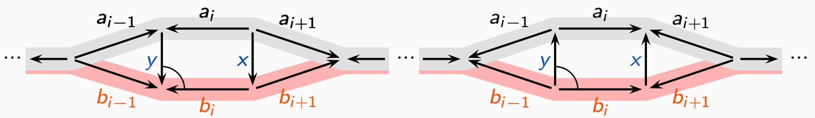

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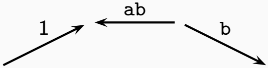

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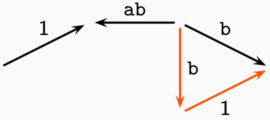

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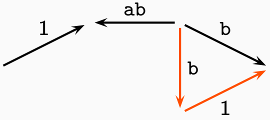

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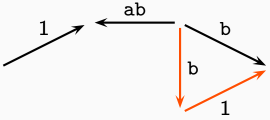

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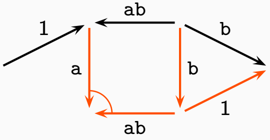

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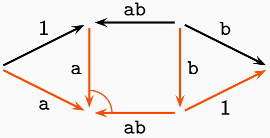

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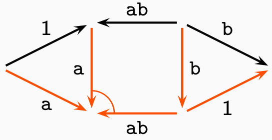

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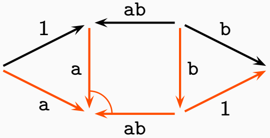

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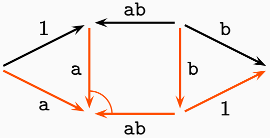

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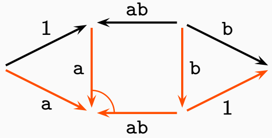

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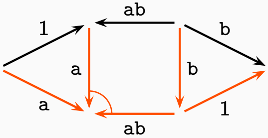

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• In this way: for every gcd-monoid M, a rewrite system $\mathcal{R}_M$ ("reduction"),

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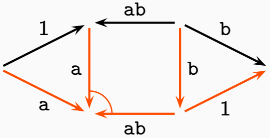

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- possible confluence (?)

• In this way: for every gcd-monoid M, a rewrite system $\mathcal{R}_M$ ("reduction"), walking among the various multifractions that represent each given element of $\mathcal{U}(M)$.

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Plan:

- 1. Reduction of multifractions
  - The enveloping group of a monoid
  - Free reduction
  - A two-step extension: (i) division, (ii) reduction

- 2. Artin–Tits monoids I
  - The FC case: two theorems
  - The general case: three conjectures

- 3. Interval monoids (joint with F. Wehrung)
  - The interval monoid of a poset
  - Examples and counter-examples

- 4. Artin–Tits monoids II (joint with D. Holt and S. Rees)
  - Padded reduction
  - The sufficiently large case

- An Artin-Tits monoid:

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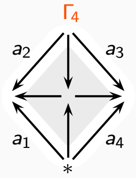

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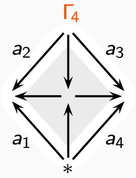

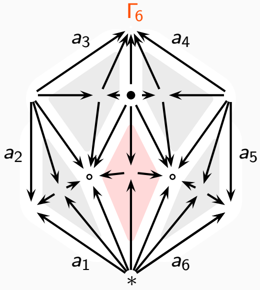

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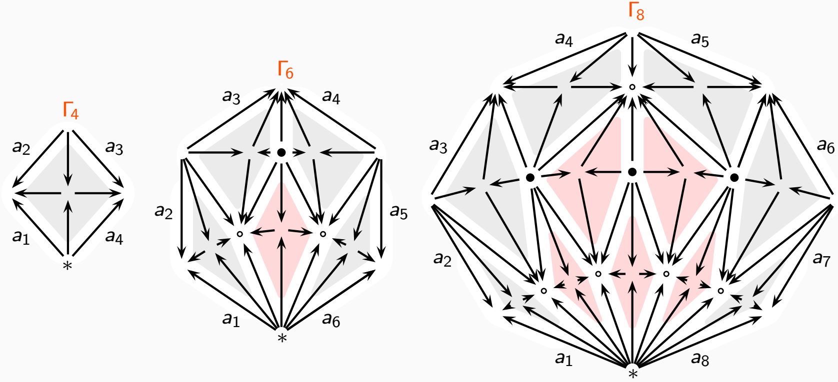

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- Proposition: (i) *If $\mathcal{R}_M$ is convergent, then it is semiconvergent.*
  (ii) *If M is a strongly noetherian gcd-monoid with finitely many basic elements and $\mathcal{R}_M$ is semiconvergent, then the word problem for $\mathcal{U}(M)$ is decidable.*

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- Finite approximation : n-semiconvergence := semiconvergence
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& \Rightarrow & 1/\mathtt{c}/\mathtt{bcb}/\mathtt{bc}/1/1 & \text{via } R_{1,\mathtt{ac}} \\
& \Rightarrow & \mathtt{bc}/1/1/\mathtt{bc}/1/1 & \text{via } R_{2,\mathtt{cbc}} \\
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\end{array}$$

- Finite approximation : n-semiconvergence := semiconvergence restricted to depth n multifractions;
  - $\mathcal{R}_M$ is 2-semiconvergent iff $M \hookrightarrow \mathcal{U}(M)$,

- Conjecture A: *For every Artin-Tits monoid M, the system $\mathcal{R}_M$ is semiconvergent.*

  - Would imply the decidability of the word problem for AT groups.
  - Similarity with the Dehn algorithm: no introduction of pairs ss^{-1} or $s^{-1}s$.
  - Supported by massive computer experiments.

- Example: type $\widetilde{\mathrm{A}}_2$: $\langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{aba} = \mathtt{bab}, \mathtt{bcb} = \mathtt{cbc}, \mathtt{cac} = \mathtt{aca} \rangle^+$
  - We have seen $\mathtt{ac/ca/ba} \Longleftarrow \mathtt{1/c/aba} \Longrightarrow \mathtt{bc/cb/ab}$
  - The quotient $\mathtt{ac/ca/ba/ab/cb/bc}$ represents 1 in $\mathcal{U}(M)$, hence should reduce to $\underline{1}$:

$$
\begin{array}{llll}
\mathtt{ac/ca/ba/ab/cb/bc} & \Rightarrow & \mathtt{ac/cac/b/1/cb/bc} & \text{via } R_{3,\mathtt{ab}} \\
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- Remember: for FC type, $\underline{a}$ represents 1 implies $\underline{a}$ reduces to $\underline{1}$ by the universal recipe.

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- Question (McCammond): Does Conjecture $\mathrm{B}^{\mathrm{geom}}$ (hence B) imply that $\mathcal{U}(M)$ satisfies the $K(\pi, 1)$-conjecture?

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  - True for FC type, with $\nabla\underline{a} := \mathrm{red}(\underline{a})$.

Plan:

- 1. Reduction of multifractions
  - The enveloping group of a monoid
  - Free reduction
  - A two-step extension: (i) division, (ii) reduction

- 2. Artin–Tits monoids I
  - The FC case: two theorems
  - The general case: three conjectures

- 3. Interval monoids (joint with F. Wehrung)
  - The interval monoid of a poset
  - Examples and counter-examples

- 4. Artin–Tits monoids II (joint with D. Holt and S. Rees)
  - Padded reduction
  - The sufficiently large case

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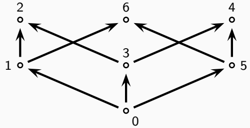

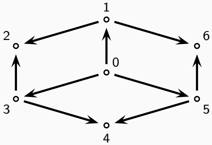

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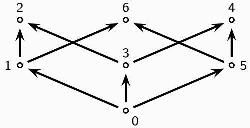
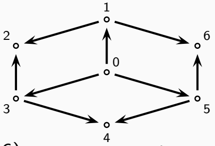
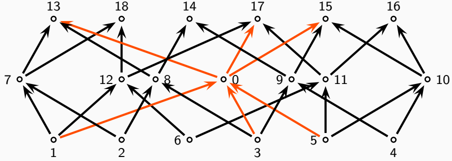
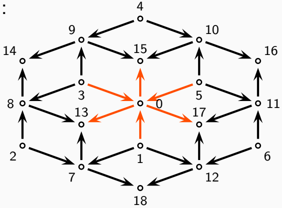




For (ii) consider the interval monoid of (here $n = 6$):





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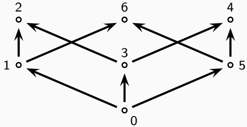

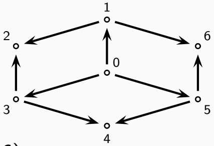

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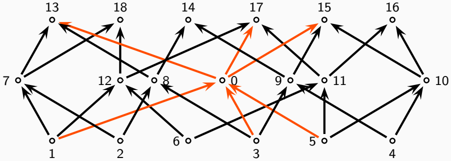

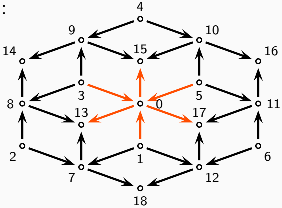

  - ▶ a necklace of n connected diamonds,

    plus a central cross connecting each other extremal vertex. □

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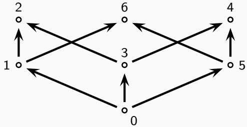
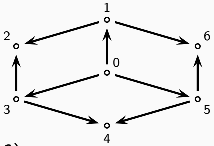
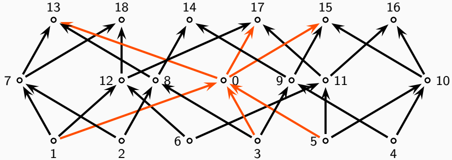
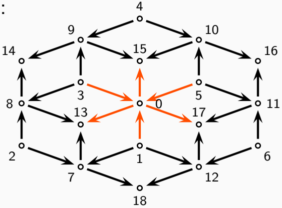




For (ii) consider the interval monoid of (here $n = 6$):





  - a necklace of n connected diamonds,
    plus a central cross connecting each other extremal vertex. $\square$

- The monoids $\mathrm{Int}(P)$ are (very) far from Artin-Tits monoids

- Proposition (D.-Wehrung): *There exist noetherian gcd-monoids such that*
  (i) $\mathcal{R}_M$ *is semiconvergent but not convergent,*
  (ii) $\mathcal{R}_M$ *is n'-semiconvergent for $n' < n$ but not n-semiconvergent.*

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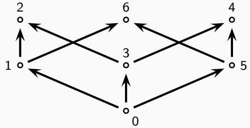

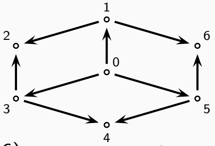

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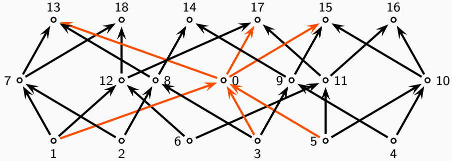

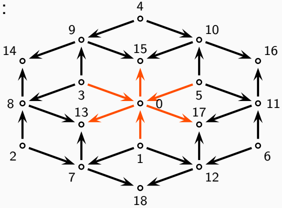

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- The monoids $\mathrm{Int}(P)$ are (very) far from Artin-Tits monoids
  - A proof of the conjectures must require specific "non-Garside" arguments.

Plan:

- 1. Reduction of multifractions
  - The enveloping group of a monoid
  - Free reduction
  - A two-step extension: (i) division, (ii) reduction
- 2. Artin–Tits monoids I
  - The FC case: two theorems
  - The general case: three conjectures
- 3. Interval monoids (joint with F. Wehrung)
  - The interval monoid of a poset
  - Examples and counter-examples
- 4. Artin–Tits monoids II (joint with D. Holt and S. Rees)
  - Padded reduction
  - The sufficiently large case

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• The "first" open case (neither FC nor sufficiently large):

$$\langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} \mid \mathtt{aba} = \mathtt{bab}, \mathtt{aca} = \mathtt{cac}, \mathtt{bcb} = \mathtt{cbc}, \mathtt{ada} = \mathtt{dad}, \mathtt{bdb} = \mathtt{dbd}, \mathtt{cd} = \mathtt{dc} \rangle^+.$$

- P. Dehornoy, *Multifraction reduction I: The 3-Ore case and Artin-Tits groups of type FC* J. Combinat. Algebra 1 (2017) 185-228, arXiv:1606.08991
- P. Dehornoy, *Multifraction reduction II: Conjectures for Artin-Tits groups* J. Combinat. Algebra 1 (2017) 229-287, arXiv:1606.08995
- P. Dehornoy & F. Wehrung, *Multifraction reduction III: The case of interval monoids* J. Combinat. Algebra1 (2017) 341-370, arXiv:1606.09018

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• P. Dehornoy, *MoKa ("monoid calculus")*, MacOS/Docker executable binaries
`www.math.unicaen.fr/∼dehornoy/programs/`.