Multifraction reduction
and the Word Problem for Artin-Tits groups

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- A new approach to the Word Problem for Artin-Tits groups (and other groups),
  - based on a rewrite system extending free reduction,
  - reminiscent of the Dehn algorithm for hyperbolic groups,
  - proved in particular cases, conjectured in the general case.

Plan:

- 1. Reduction of multifractions
  - The enveloping group of a monoid
  - Free reduction
  - A two-step extension: (i) division, (ii) reduction
- 2. Artin–Tits monoids I
  - The FC case: two theorems
  - The general case: three conjectures
- 3. Interval monoids (joint with F. Wehrung)
  - The interval monoid of a poset
  - Examples and counter-examples
- 4. Artin–Tits monoids II (joint with D. Holt and S. Rees)
  - Padded reduction
  - The sufficiently large case

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- For every monoid M, there exists a unique group $\mathcal{U}(M)$ and a unique morphism $\phi : M \to \mathcal{U}(M)$ s.t. every morphism from M to a group factors through ϕ.
  If $M = \langle S \mid R \rangle^+$, then $\mathcal{U}(M) = \langle S \mid R \rangle$.
  - Caution! Even if M is cancellative, ϕ need not be injective: Mal'cev conditions.

- Theorem (Ore, 1933): *If M is cancellative and satisfies the 2-Ore condition, then M embeds in $\mathcal{U}(M)$ and every element of $\mathcal{U}(M)$ is represented as ab^{-1} with a, b in M.*
  - "group of (right) fractions for M"

- Say that a left-divides b, or b is a right-multiple of a, if $\exists x\,(ax = b)$. $\leftarrow a \leqslant b$
  - 2-Ore condition: any two elements admit a common right-multiple.

- Definition: A gcd-monoid is a cancellative monoid, in which 1 is the only invertible element and any two elements admit a left- and a right-gcd.

a $\leqslant$- and a $\widetilde{\leqslant}$-greatest lower bound ↑ then $\leqslant$ is a partial order ↑

- Corollary: *If M is a gcd-monoid satisfying the 2-Ore condition, then M embeds in $\mathcal{U}(M)$ and every element of $\mathcal{U}(M)$ is represented by a unique irreducible fraction.*

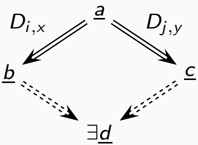
↑ ab^{-1} with $a, b \in M$ and right-gcd$(a, b) = 1$

- Example: $M = B_n^+$, the n-strand braid monoid; more generally, every Garside monoid.

- When the 2-Ore condition fails (no common multiples), no fractional expression.

- Example: $M = F^+$, a free monoid; then M embeds in $\mathcal{U}(M)$, a free group;
  - No fractional expression for the elements of $\mathcal{U}(M)$,
  - But: unique expression $a_1 a_2^{-1} a_3 a_4^{-1} \cdots$ with $a_1, a_2, \ldots$ in M and
    for i odd: a_i and a_{i+1} do not finish with the same letter,
    for i even: a_i and a_{i+1} do not begin with the same letter.
    - a "freely reduced word"

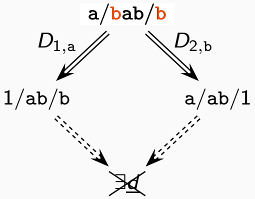
- Proof: (easy) Introduce rewrite rules on finite sequences of positive words:
  - rule $D_{i,x} := \begin{cases} \text{for } i \text{ odd, delete } x \text{ at the end of } a_i \text{ and } a_{i+1} \text{ (if possible...)}, \\ \text{for } i \text{ even, delete } x \text{ at the beginning of } a_i \text{ and } a_{i+1} \text{ (if possible...)}. \end{cases}$
  - Then the system of all rules $D_{i,x}$ is (locally) confluent:



  - Every sequence $\underline{a}$ rewrites into a unique irreducible sequence ("convergence"). □

- When M is not free, the rewrite rule $D_{i,x}$ can still be given a meaning:
  - no first or last letter,
  - but left- and right-divisors: $x \leqslant a$ means "x is a possible beginning of a".
  - rule $D_{i,x} := \begin{cases} \text{for } i \text{ odd, right-divide } a_i \text{ and } a_{i+1} \text{ by } x \text{ (if possible...)}, \\ \text{for } i \text{ even, left-divide } a_i \text{ and } a_{i+1} \text{ by } x \text{ (if possible...)}. \end{cases}$

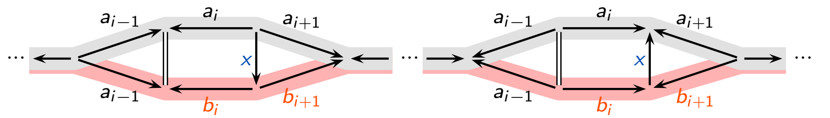
- Example: $M = B_3^+ = \langle \mathtt{a}, \mathtt{b} \mid \mathtt{aba} = \mathtt{bab} \rangle^+$;
  - start with the sequence $(\mathtt{a}, \mathtt{aba}, \mathtt{b})$, better written $\mathtt{a}/\mathtt{aba}/\mathtt{b}$ ("multifraction")



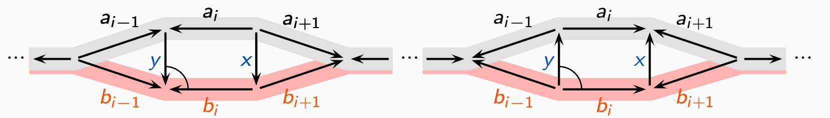
  - no hope of confluence...
  - consider more general rewrite rules.

- Diagrammatic representation of elements of M: $\xrightarrow{a} \;\mapsto a$, and multifractions:
  $\xrightarrow{a_1}\xleftarrow{a_2}\xrightarrow{a_3}\cdots \;\mapsto\; a_1/a_2/a_3/\ldots \;\mapsto\; \phi(a_1)\phi(a_2)^{-1}\phi(a_3)\ldots$ in $\mathcal{U}(M)$.
  - Then: commutative diagram $\leftrightarrow$ equality in $\mathcal{U}(M)$.

- Diagram for $D_{i,x}$ (division by x at level i): declare $\underline{a} \bullet D_{i,x} = \underline{b}$ for



- Relax "x divides a_i" to "lcm(x, a_i) exists": declare $\underline{a} \bullet R_{i,x} = \underline{b}$ for

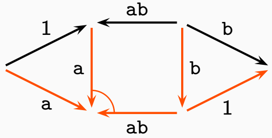


- Definition: "$\underline{b}$ obtained from $\underline{a}$ by reducing x at level i":
  divide a_{i+1} by $\mathbf{x}$, push $\mathbf{x}$ through a_i using lcm, multiply a_{i-1} by the remainder $\mathbf{y}$.

- $\underline{a} \bullet D_{i,x}$ is defined if x divides both a_i and a_{i+1};
- $\underline{a} \bullet R_{i,x}$ is defined if x divides a_{i+1}, and x and a_i have a common multiple.

• Fact: $\underline{b} = \underline{a} \bullet R_{i,x}$ implies that $\underline{a}$ and $\underline{b}$ represent the same element in $\mathcal{U}(M)$.

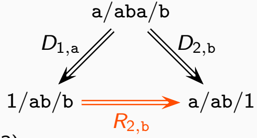
• Example: $M = B_3^+$ with $1/\mathtt{ab}/\mathtt{b}$:



b and ab admit a common multiple

- we can push b through ab:
- $\mathtt{a}/\mathtt{ab}/1 = 1/\mathtt{ab}/\mathtt{b} \bullet R_{2,\mathtt{b}}$

and now



- possible confluence (?)

• In this way: for every gcd-monoid M, a rewrite system $\mathcal{R}_M$ ("reduction"), walking among the various multifractions that represent each given element of $\mathcal{U}(M)$.

• Theorem: (i) *If M is a* **noetherian** *gcd-monoid satisfying the* 3-**Ore condition**, *then M embeds in $\mathcal{U}(M)$ and $\mathcal{R}_M$ is convergent: every element of $\mathcal{U}(G)$ is represented by a unique $\mathcal{R}_M$-irreducible multifraction.*

(ii) *If, moreover, M is* **strongly noetherian** *and has finitely many* **basic** *elements, then the word problem for $\mathcal{U}(M)$ is decidable.*

- M is noetherian: no infinite descending sequence for left- and right-divisibility.
- M is strongly noetherian: exists a pseudo-length function on M. ($\Rightarrow$ noetherian)
- M satisfies the 3-Ore condition: three elements that pairwise admit a common multiple admit a global one. (2-Ore $\Rightarrow$ 3-Ore)
- right-basic elements: obtained from atoms repeatedly using the right-complement operation: $(x, y) \mapsto x'$ s.t. $yx' = \text{right-lcm}(x, y)$.

• Proof: (i) The rewrite system $\mathcal{R}_M$ is convergent:
- noetherianity of M ensures termination;
- the 3-Ore condition ensures confluence.

(ii) Finitely many basic elements provides an upper bound for possible common multiples, ensuring that $\Rightarrow$ is decidable. □

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- An Artin-Tits monoid: $\langle S \mid R \rangle^+$ such that, for all s, t in S,
  there is at most one relation $s... = t...$ in R and, if so, the relation has the form
  $stst... = tsts...$, both terms of same length (a "braid relation").

- Proposition (Brieskorn–Saito, 1971): *An Artin-Tits monoid satisfies the 2-Ore condition iff it of spherical type.*
  ↑
  adding $s^2 = 1$ for every s in S yields a finite Coxeter group
  - "Garside theory"

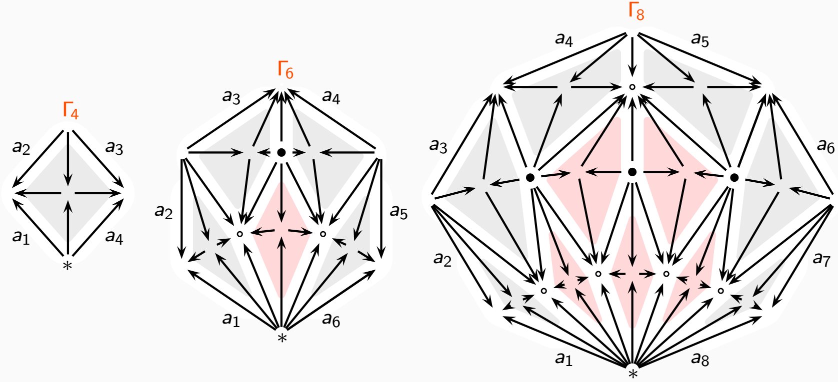
- Proposition: *An Artin-Tits monoid satisfies the 3-Ore condition iff it of FC ("flag complex") type.*
  ↑
  if $\forall s, t \in S' \subseteq S \; \exists s... = t...$ in R, then $\langle S' \rangle$ is spherical

- Theorem 1: *If M is a FC-type Artin-Tits monoid, then every element of the group $\mathcal{U}(M)$ is represented by a unique $\mathcal{R}_M$-irreducible multifraction.*

  - a new normal form
    ($\neq$ the Niblo–Reeves n.f. deduced from the action on a CAT(0)-complex)

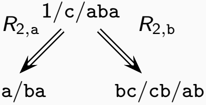
• <u>Theorem 2</u>: *For every n, there exists a universal sequence of integers $U(n)$ such that, whenever M is a FC-type Artin-Tits monoid and $\underline{a}$ is any depth n multifraction representing 1 in $\mathcal{U}(M)$, then $\underline{a}$ reduces to $\underline{1}$ by maximal reductions at levels $U(n)$.*

- Example: $U(8) = (1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 1, 2, 3, 1)$.
- Works for every gcd-monoid satisfying the 3-Ore condition.
- Geometric interpretation: existence of a universal van Kampen diagram Γ_n:

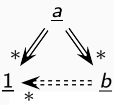


- Every AT-monoid satisfies some of the assumptions:
  - is strongly noetherian (relations preserve the length of words),
  - has finitely many basic elements (D.-Dyer-Hohlweg, 2015),
  - but does not necessarily satisfy the 3-Ore condition, i.e., is of FC-type...

- Example: type $\widetilde{A}_2$: $\langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{aba} = \mathtt{bab}, \mathtt{bcb} = \mathtt{cbc}, \mathtt{cac} = \mathtt{aca} \rangle^+$
  - the elements a, b, c pairwise admit common multiples, but no global multiple
  - the rewrite system $\mathcal{R}_M$ is not confluent:



- Definition: $\mathcal{R}_M$ is semiconvergent if
  $\underline{a}$ represents 1 in $\mathcal{U}(M)$ iff $\underline{a} \Rightarrow^* \underline{1}$.
  - Equivalently: conjunction of $\underline{a} \Rightarrow^* \underline{1}$ and $\underline{a} \Rightarrow^* \underline{b}$ implies $\underline{b} \Rightarrow^* \underline{1}$:



- Proposition: (i) *If $\mathcal{R}_M$ is convergent, then it is semiconvergent.*
  (ii) *If M is a strongly noetherian gcd-monoid with finitely many basic elements and $\mathcal{R}_M$ is semiconvergent, then the word problem for $\mathcal{U}(M)$ is decidable.*

- Conjecture A: *For every Artin-Tits monoid M, the system $\mathcal{R}_M$ is semiconvergent.*

  - Would imply the decidability of the word problem for AT groups.
  - Similarity with the Dehn algorithm: no introduction of pairs ss^{-1} or $s^{-1}s$.
  - Supported by massive computer experiments.

- Example: type $\widetilde{A}_2$: $\langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{aba} = \mathtt{bab}, \mathtt{bcb} = \mathtt{cbc}, \mathtt{cac} = \mathtt{aca} \rangle^+$
  - We have seen $\mathtt{ac/ca/ba} \Longleftarrow \mathtt{1/c/aba} \Longrightarrow \mathtt{bc/cb/ab}$
  - The quotient $\mathtt{ac/ca/ba/ab/cb/bc}$ represents 1 in $\mathcal{U}(M)$, hence should reduce to $\underline{1}$:

$$\begin{array}{lcll}
\mathtt{ac/ca/ba/ab/cb/bc} & \Rightarrow & \mathtt{ac/cac/b/1/cb/bc} & \text{via } R_{3,\mathtt{ab}} \\
 & \Rightarrow & \mathtt{ac/cac/bcb/1/1/bc} & \text{via } R_{4,\mathtt{cb}} \\
 & \Rightarrow & \mathtt{ac/cac/bcb/bc/1/1} & \text{via } R_{5,\mathtt{bc}} \\
 & \Rightarrow & \mathtt{1/c/bcb/bc/1/1} & \text{via } R_{1,\mathtt{ac}} \\
 & \Rightarrow & \mathtt{bc/1/1/bc/1/1} & \text{via } R_{2,\mathtt{cbc}} \\
 & \Rightarrow & \mathtt{bc/bc/1/1/1/1} & \text{via } R_{3,\mathtt{bc}} \\
 & \Rightarrow & \mathtt{1/1/1/1/1/1} & \text{via } R_{1,\mathtt{bc}}
\end{array}$$

- Finite approximation : n-semiconvergence := semiconvergence restricted to depth n multifractions;
  - $\mathcal{R}_M$ is 2-semiconvergent iff $M \hookrightarrow \mathcal{U}(M)$, which is true (L. Paris, 2001).

- Remember: for FC type, $\underline{a}$ represents 1 implies $\underline{a}$ reduces to $\underline{1}$ by the universal recipe.

- Conjecture B: *For every Artin-Tits monoid M, if a multifraction $\underline{a}$ represents 1 in $\mathcal{U}(M)$, then $\underline{a}$ reduces to $\underline{1}$ by the universal recipe.*

  - Implies Conjecture A, hence the decidability of the word problem for the group.
  - More precise than A, hence (maybe) more difficult to prove, but easier to test.
  - For depth 4 multifractions, Conjecture A equivalent to Conjecture B.
  - Implies the existence of universal van Kampen diagrams.
    - call this property "Conjecture $\mathrm{B}^{\mathrm{geom}}$".

- Question: Does Conjecture $\mathrm{B}^{\mathrm{geom}}$ (hence B) imply that $\mathcal{U}(M)$ is torsion-free?
- Question (McCammond): Does Conjecture $\mathrm{B}^{\mathrm{geom}}$ (hence B) imply that $\mathcal{U}(M)$ satisfies the $K(\pi, 1)$-conjecture?

- So far: left reduction: pushing factors to the left
  - Possible to reverse the definition and push factors to the right
  - A system $\widetilde{\mathcal{R}}_M$ ("right reduction"), with entirely symmetric properties.

- Definition: The system $\mathcal{R}_M$ is cross-confluent if

- Conjecture C: *For every Artin-Tits monoid M, the system $\mathcal{R}_M$ is cross-confluent.*

  - Implies Conjecture A, hence the decidability of the word problem for the group.
  - Local version (no * = one step of reduction) is true.
  - Uniform (stronger) version: $\forall \underline{a}\ \exists \nabla\underline{a}\ \forall \underline{b}\ (\underline{a} \overset{\sim}{\Longrightarrow}{}^* \underline{b}$ implies $\underline{b} \Longrightarrow^* \nabla\underline{a})$.
  - True for FC type, with $\nabla\underline{a} := \mathrm{red}(\underline{a})$.

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- Definition (F.Wehrung): For $(P, \leqslant)$ a poset, the interval monoid of P is

$$\mathrm{Int}(P) := \langle \{[x, y] \mid x < y \in P\} \mid \{[x, y][y, z] = [x, z] \mid x < y < z \in P\} \rangle^{+}.$$

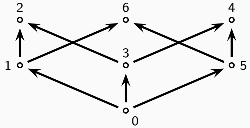
the intervals of P

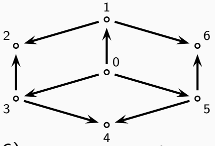
- Proposition (D.–Wehrung) *If M is the interval monoid of a finite poset P, and M is a gcd-monoid (which is effectively checkable when P is finite), there is a notion of a reducible simple circuit in P such that*
  - *If all simple circuits of P are reducible, then $\mathcal{R}_M$ is semiconvergent;*
  - *If all length $\leqslant n$ simple circuits of P are reducible, then $\mathcal{R}_M$ is n-semiconvergent.*

- Checkable conditions for P finite: finitely many simple circuits.

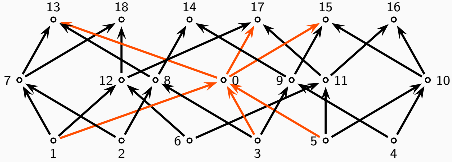
- <u>Proposition</u> (D.-Wehrung): *There exist noetherian gcd-monoids such that*
  (i) $\mathcal{R}_M$ *is semiconvergent but not convergent,*
  (ii) $\mathcal{R}_M$ *is* n'*-semiconvergent for* $n' < n$ *but not* n*-semiconvergent.*

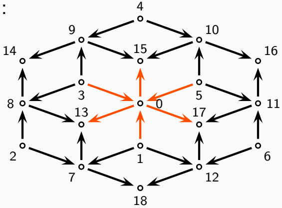
- Proof: For (i), consider the interval monoid of





For (ii) consider the interval monoid of (here $n = 6$):





  - a necklace of n connected diamonds,
    plus a central cross connecting each other extremal vertex. □

- The monoids $\mathrm{Int}(P)$ are (very) far from Artin-Tits monoids
  - A proof of the conjectures must require specific "non-Garside" arguments.

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- Recall: $\mathcal{R}_M$ semiconvergent if "$\underline{a}$ represents 1 in $\mathcal{U}(M)$ implies $\underline{a} \Rightarrow^* \underline{1}$".

- Definition: $\mathcal{R}_M$ is semiconvergent up to f-padding if
"$\underline{a}$ represents 1 in $\mathcal{U}(M)$ implies $\underline{1}_{2f(\underline{a})}/\underline{a} \Rightarrow^* \underline{1}$".
↑ adding $2f(\underline{a})$ "dummy 1s" at the beginning of $\underline{a}$
   - $\underline{a} \Rightarrow^* \underline{1}$ implies $\underline{1}_{2m}/\underline{a} \Rightarrow^* \underline{1}$,
   - but, conversely, adding dummy 1s ("padding") allows for more reductions,
   - so $\underline{1}_{2m}/\underline{a} \Rightarrow^* \underline{1}$ need not imply $\underline{a} \Rightarrow^* \underline{1}$.

- Proposition: *If M is a strongly noetherian gcd-monoid with finitely many basic elements and $\mathcal{R}_M$ is semiconvergent up to f-padding for some Turing-computable f, then the word problem for $\mathcal{U}(M)$ is decidable.*

- Conjecture A^{padded}: *For every Artin-Tits monoid M, the system $\mathcal{R}_M$ is semiconvergent up to Turing-computable padding.*
   - Would imply the decidability of the word problem for AT groups.

• An Artin–Tits monoid is said to be of sufficiently large type if,
in any triangle in the associated Coxeter diagram,

- either no edge has label 2 ("large type"),
- or all three edges have label 2 ("right-angled"),
- or at least one edge has label ∞ ("free").

• Theorem (D.–Holt–Rees): *If M is an AT-monoid of sufficiently large type, then $\mathcal{R}_M$ is semiconvergent up to a quadratic padding.*

• The "first" open case (neither FC nor sufficiently large):

$$\langle \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \mid \mathrm{aba} = \mathrm{bab}, \mathrm{aca} = \mathrm{cac}, \mathrm{bcb} = \mathrm{cbc}, \mathrm{ada} = \mathrm{dad}, \mathrm{bdb} = \mathrm{dbd}, \mathrm{cd} = \mathrm{dc} \rangle^{+}.$$

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- P. Dehornoy, *Multifraction reduction II: Conjectures for Artin-Tits groups* J. Combinat. Algebra 1 (2017) 229-287, arXiv:1606.08995

- P. Dehornoy & F. Wehrung, *Multifraction reduction III: The case of interval monoids* J. Combinat. Algebra1 (2017) 341-370, arXiv:1606.09018

- P. Dehornoy, D. Holt, & S. Rees, *Multifraction reduction IV: Padding and Artin-Tits groups of sufficiently large type* J. Pure Appl. Algebra, to appear, arXiv:1701.06413

- F. Wehrung, *Gcd-monoids arising from homotopy groupoids* hal:01338106

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