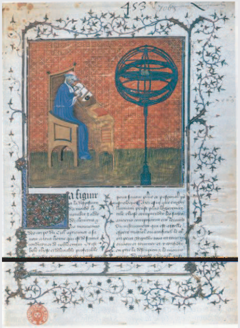
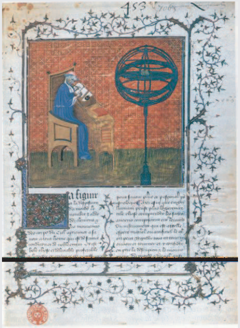
Set Theory fifty years after Cohen



Set Theory fifty years after Cohen

Patrick Dehornoy

Laboratoire de Mathématiques
Nicolas Oresme, Université de Caen



Set Theory fifty years after Cohen

Patrick Dehornoy

Laboratoire de Mathématiques
Nicolas Oresme, Université de Caen

São Carlos e São Paulo, agosto 2016

Abstract

Abstract

- Cohen's work is not the end of History.

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.
- New types of applications of Set Theory appear.

Abstract

- Cohen’s work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.
- New types of applications of Set Theory appear.

Plan

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.
- New types of applications of Set Theory appear.

Plan

- I. The Continuum Problem up to Cohen

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.
- New types of applications of Set Theory appear.

Plan

- I. The Continuum Problem up to Cohen
- II. What does discovering new true axioms mean?

Abstract

- Cohen's work is not the end of History.
- Today (much) more is known about (sets and) infinities,
  and there is a reasonable hope that the Continuum Problem will be solved.
- New types of applications of Set Theory appear.

Plan

- I. The Continuum Problem up to Cohen
- II. What does discovering new true axioms mean?
- III. An application of a new type: Laver tables

I. The Continuum Problem up to Cohen

- Theorem (Cantor, 1873): *There exist*
  at least two non-equivalent infinities.

- Theorem (Cantor, 1873): *There exist* *at least two* *non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist* *infinitely many* *non-equivalent infinities,*

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Theorem (Cantor, 1873): *There exist* *at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist* *infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - card($\mathbb{N}$) $= \aleph_0$,
  - card($\mathbb{R}$)

- Theorem (Cantor, 1873): *There exist* *at least two* *non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist* *infinitely many* *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,
  - $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathfrak{P}(\mathbb{N}))$

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - card($\mathbb{N}$) $= \aleph_0$,
  - card($\mathbb{R}$) $=$ card($\mathfrak{P}(\mathbb{N})$) $= 2^{\aleph_0}$

- Theorem (Cantor, 1873): *There exist* *at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist* *infinitely many non-equivalent infinities,* *which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,
  - $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} > \mathrm{card}(\mathbb{N})$.

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts.
  - card$(\mathbb{N}) = \aleph_0$,
  - card$(\mathbb{R}) =$ card$(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} >$ card$(\mathbb{N})$.

- Question (**Continuum Problem**):

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,
  - $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} > \mathrm{card}(\mathbb{N})$.

- Question (**Continuum Problem**): For which α does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - card$(\mathbb{N}) = \aleph_0$,
  - card$(\mathbb{R})$ = card$(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} >$ card$(\mathbb{N})$.

- Question (**Continuum Problem**): For which α does card$(\mathbb{R}) = \aleph_\alpha$ hold?

- Conjecture (**Continuum Hypothesis**, Cantor, 1879):

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts. - card$(\mathbb{N}) = \aleph_0$,
  - card$(\mathbb{R})$ = card$(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} >$ card$(\mathbb{N})$.

- Question (**Continuum Problem**): For which α does card$(\mathbb{R}) = \aleph_\alpha$ hold?

- Conjecture (**Continuum Hypothesis**, Cantor, 1879): card$(\mathbb{R}) = \aleph_1$.

- Theorem (Cantor, 1873): *There exist at least two non-equivalent infinities.*

- Theorem (Cantor, 1880's): *There exist infinitely many non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots.$$

- Facts. - $\mathrm{card}(\mathbb{N}) = \aleph_0$,
  - $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0} > \mathrm{card}(\mathbb{N})$.

- Question (**Continuum Problem**): For which α does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?

- Conjecture (**Continuum Hypothesis**, Cantor, 1879): $\mathrm{card}(\mathbb{R}) = \aleph_1$.
  - ▶ Equivalently: every uncountable set of reals has the cardinality of $\mathbb{R}$.

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH.

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH*.*

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH*.*

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH*.*

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH.

↑ Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH.

… and then no progress for 70 years.

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH*.*

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH*.*

... and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, ...)

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH*.*

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH*.*

... and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH.

↑
Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH.

… and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, …) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - **Consensus**: "*We agree that these properties express our current intuition of sets.*"

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH*.*

↑ Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH*.*

… and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, …) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - **Consensus**: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future…)*

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH.

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH.

… and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, …)
and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - **Consensus**: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future…)*

- First question:

- Theorem (Cantor–Bendixson, 1883): *Closed sets satisfy* CH.

↑

Every closed set of reals either is countable or has the cardinality of $\mathbb{R}$.

- Theorem (Alexandroff, 1916): *Borel sets satisfy* CH.

… and then no progress for 70 years.

- In the meanwhile, formalization of First Order logic (Frege, Russell, …) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - **Consensus**: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future…*)

- First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,*

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* $\neg$CH *cannot be proved from* ZF.

$\uparrow$
negation of

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,*
  $\neg$CH *cannot be proved from* ZF*.*

↑
negation of

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,*
  $\neg$CH *cannot be proved from* ZF.

$\uparrow$
negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,*
  CH *cannot be proved from* ZF.

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,*
  $\neg$CH *cannot be proved from* ZF.

↑
negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,*
  CH *cannot be proved from* ZF.

- Conclusion:

Two major results

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* $\neg$CH *cannot be proved from* ZF.

↑
negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

- Conclusion: ZF is incomplete.

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* $\neg$CH *cannot be proved from* ZF.

$\uparrow$
negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

- Conclusion: ZF is incomplete.
  - Discover further properties of sets, and adopt an **extended list** of axioms!

Two major results

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* $\neg$CH *cannot be proved from* ZF.

↑
negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

- Conclusion: ZF is incomplete.
  - Discover further properties of sets, and adopt an **extended list** of axioms!
  - How to recognize that an axiom is true? (What does this mean?)

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* ¬CH *cannot be proved from* ZF.

↑ negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

- Conclusion: ZF is incomplete.
  - Discover further properties of sets, and adopt an **extended list** of axioms!
  - How to recognize that an axiom is true? (What does this mean?)

  Example: CH **may** be taken as an additional axiom, but **not** a good idea...

II. What does discovering new true axioms mean?

- Which new axioms?

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):

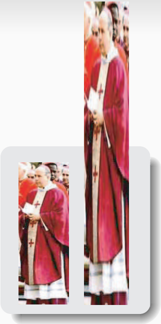
- Which new axioms?

- From 1930's, axioms of large cardinal (LC):

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation

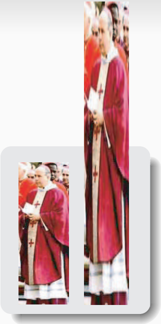
$$\frac{\text{super-infinite}}{\text{infinite}}$$



- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$



- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals,

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

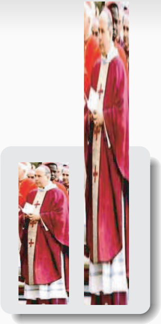
- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)

- Which new axioms?

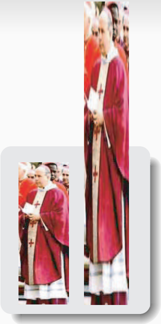
- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.



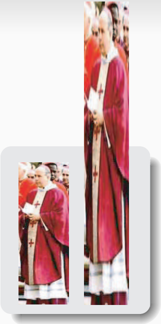
- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.



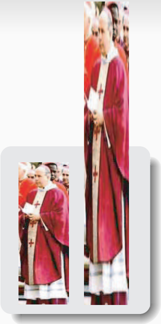
- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\underset{\substack{\uparrow \\ \text{a } \textbf{self-embedding} \text{ of } X}}{\exists j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)



- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
  $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\underset{\substack{\uparrow \\ \text{a }\textbf{self-embedding}\text{ of } X}}{\exists j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)
  - Example: No self-embedding of $\mathbb{N}$ may exist, hence $\mathbb{N}$ is not super-infinite.



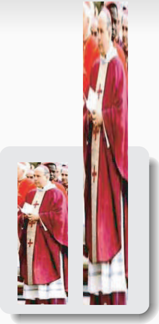
- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\exists \underset{\substack{\uparrow \\ \text{a self-embedding of } X}}{j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    a **self-embedding** of X
  - Example: No self-embedding of $\mathbb{N}$ may exist, hence $\mathbb{N}$ is not super-infinite.

- Then: LC are natural axioms



- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\exists \underset{\uparrow}{j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    a **self-embedding** of X
  - Example: No self-embedding of $\mathbb{N}$ may exist, hence $\mathbb{N}$ is not super-infinite.

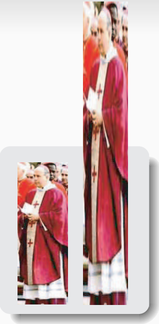
- Then: LC are natural axioms (iteration of the postulate that infinite sets exist),

- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\underset{\substack{\uparrow \\ \text{a self-embedding of } X}}{\exists j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)
  - Example: No self-embedding of $\mathbb{N}$ may exist, hence $\mathbb{N}$ is not super-infinite.

- Then: LC are natural axioms (iteration of the postulate that infinite sets exist), but no evidence that they are true, or just useful



- Which new axioms?

- From 1930's, axioms of large cardinal (LC):
  - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - **inaccessible** cardinals, **measurable** cardinals, etc.

- Principle: self-similar implies large
  - X **infinite**: $\exists j : X \to X$ (j injective not bijective)
  - X **super**-infinite: $\underset{\substack{\uparrow \\ \text{a \textbf{self-embedding} of } X}}{\exists j} : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)
  - Example: No self-embedding of $\mathbb{N}$ may exist, hence $\mathbb{N}$ is not super-infinite.

- Then: LC are natural axioms (iteration of the postulate that infinite sets exist), but no evidence that they are true, or just useful (no connection with ordinary objects).

- **Definition**: For $A \subseteq \mathbb{R}$,

- <u>Definition</u>: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

  I
  II

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{ll} \mathrm{I} & a_1 \\ \mathrm{II} & \end{array}$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lll} \text{I} & a_1 & \\ \text{II} & & a_2 \end{array}$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0, 1\}$-game $\mathbf{G_A}$:

$$\begin{array}{llll} \mathrm{I} & a_1 & & a_3 \\ \mathrm{II} & & a_2 & \end{array}$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccc} \text{I} & a_1 & & a_3 & \\ \text{II} & & a_2 & & a_4 \end{array}$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc}
\mathrm{I} & a_1 & & a_3 & & \ldots & & \\
\mathrm{II} & & a_2 & & a_4 & & \ldots &
\end{array}$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lllllll} \mathrm{I} & a_1 & & a_3 & & \dots & \\ \mathrm{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc} \text{I} & a_1 & & a_3 & & \ldots & \\ \text{II} & & a_2 & & a_4 & & \ldots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

I	a_1		a_3		$\dots$	
II		a_2		a_4		$\dots$

where I **wins** if the real $[0, a_1 a_2 \ldots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lllllll} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ... ([0, a_1 a_2 ...]_2 \in A)$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ...([0, a_1 a_2 ...]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 ...([0, a_1 a_2 ...]_2 \notin A),$$

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ...([0, a_1 a_2 ...]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 ...([0, a_1 a_2 ...]_2 \notin A),$$

and a model for many properties:

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \mathrm{I} & a_1 & & a_3 & & \dots & \\ \mathrm{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ...([0, a_1 a_2 ...]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 ...([0, a_1 a_2 ...]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \ldots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \ldots ([0, a_1 a_2 \ldots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \ldots ([0, a_1 a_2 \ldots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.

A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 ...]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 ... ([0, a_1 a_2 ...]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 ... ([0, a_1 a_2 ...]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \text{I} & a_1 & & a_3 & & \ldots & \\ \text{II} & & a_2 & & a_4 & & \ldots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \ldots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \ldots ([0, a_1 a_2 \ldots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \ldots ([0, a_1 a_2 \ldots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:
  - All closed sets are determined (Gale–Stewart, 1962),

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \mathrm{I} & a_1 & & a_3 & & \dots & \\ \mathrm{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:
  - All closed sets are determined (Gale–Stewart, 1962),
  - All Borel sets are determined (Martin, 1975).

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lllllll} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:
  - All closed sets are determined (Gale–Stewart, 1962),
  - All Borel sets are determined (Martin, 1975).
  - “All sets are determined” contradicts AC (Mycielski–Steinhaus, 1962),

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lcccccc} \mathrm{I} & a_1 & & a_3 & & \dots & \\ \mathrm{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \text{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:
  - All closed sets are determined (Gale–Stewart, 1962),
  - All Borel sets are determined (Martin, 1975).
  - "All sets are determined" contradicts AC (Mycielski–Steinhaus, 1962),
  - "All **projective** sets are determined" unprovable from ZF ($\approx$ Gödel, 1938).

- Definition: For $A \subseteq \mathbb{R}$, consider the two player $\{0,1\}$-game $\mathbf{G_A}$:

$$\begin{array}{lccccccc} \text{I} & a_1 & & a_3 & & \dots & \\ \text{II} & & a_2 & & a_4 & & \dots \end{array}$$

where I **wins** if the real $[0, a_1 a_2 \dots]_2$ belongs to A.
Then A is called **determined** if one of the players has a winning strategy in G_A.

- An infinitary statement of a special type:

$$\exists a_1 \forall a_2 \exists a_3 \dots ([0, a_1 a_2 \dots]_2 \in A) \textbf{ or } \forall a_1 \exists a_2 \forall a_3 \dots ([0, a_1 a_2 \dots]_2 \notin A),$$

and a model for many properties: there exist codings $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{B}} : \mathfrak{P}(\mathbb{R}) \to \mathfrak{P}(\mathbb{R})$ s.t.
A is Lebesgue measurable iff $C_{\mathcal{L}}(A)$ is determined,
A has the Baire property iff $C_{\mathcal{B}}(A)$ is determined, etc.

- Always true for simple sets and (false) for complicated sets:
  - All closed sets are determined (Gale–Stewart, 1962),
  - All Borel sets are determined (Martin, 1975).
  - “All sets are determined” contradicts AC (Mycielski–Steinhaus, 1962),
  - “All **projective** sets are determined” unprovable from ZF ($\approx$ Gödel, 1938).

↑
closure of Borel sets under continuous image and complement

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s):

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

↑
heuristically complete

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

↑ heuristically complete

↑ no pathologies: Lebesgue measurable, etc.

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF, PD *provides a complete and satisfactory description of projective sets of reals.*

↑ heuristically complete

↑ no pathologies: Lebesgue measurable, etc.

- Example: Under ZF + PD, projective sets satisfy CH.

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

  ↑ heuristically complete ↑ no pathologies: Lebesgue measurable, etc.

  - Example: Under ZF + PD, projective sets satisfy CH.

- So: PD is useful (gives a better description of usual sets),

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

↑ heuristically complete

↑ no pathologies: Lebesgue measurable, etc.

  - Example: Under ZF + PD, projective sets satisfy CH.

- So: PD is useful (gives a better description of usual sets),
  but not natural (why consider it?),

- Definition: Axiom of **Projective Determinacy** (PD):
  "Every projective set of reals is determined".

- Propositions (Moschovakis, Kechris, ..., 1970s): *When added to* ZF,
  PD *provides a complete and satisfactory description of projective sets of reals.*

↑ heuristically complete

↑ no pathologies: Lebesgue measurable, etc.

  - Example: Under ZF $+$ PD, projective sets satisfy CH.

- So: PD is useful (gives a better description of usual sets),
  but not natural (why consider it?),
  contrary to large cardinal axioms, which are natural but (a priori) not useful.

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑
infinitely many Woodin
cardinals imply PD

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

The axiom PD is true

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and

The axiom PD is true

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). □

The axiom PD is true

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). □

- Why "**true**"?

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). ☐

- Why "**true**"?
  - ▶ Compare with the axiom of infinity:
    Evidence = (?) interiorization of a long familiarity and of practical efficiency.

The axiom PD is true

- **Theorem** (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- **Corollary** (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). □

- Why "**true**"?
  - Compare with the axiom of infinity:
    Evidence =(?) interiorization of a long familiarity and of practical efficiency.
  - (Woodin) *"The statement that* PD *is consistent is a new mathematical truth. It predicts facts about our world, for instance that in the next 1000 years there will be no contradiction discovered from PD by any means."*

- Theorem (Martin–Steel 1985, Woodin, 1987): PD *is a large cardinal axiom.*

↑ infinitely many Woodin cardinals imply PD

↑ PD (implies) infinitely many Woodin cardinals

- Corollary (Woodin): PD *is true.*

"Proof": PD is both natural (as a large cardinal axiom), and useful (as a determinacy property). ☐

- Why "**true**"?
  - ▶ Compare with the axiom of infinity:
    Evidence = (?) interiorization of a long familiarity and of practical efficiency.
  - ▶ (Woodin) *"The statement that* PD *is consistent is a new mathematical truth. It predicts facts about our world, for instance that in the next 1000 years there will be no contradiction discovered from PD by any means."*
  - ▶ New consensus: The base system for 21th century Set Theory is no longer ZF, but ZF + PD.

- Fact: CH and $\neg$CH not provable from $\mathsf{ZF} + \mathsf{PD}$:

- Fact: CH and $\neg$CH not provable from $\mathsf{ZF} + \mathsf{PD}$: description not yet complete...

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;

- Fact: CH and ¬CH not provable from $\mathrm{ZF} + \mathrm{PD}$: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

… which will necessarily entail a solution to CH.

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

(think of $\mathbb{C}$ in the world of number fields of characteristic 0)

- Fact: CH and ¬CH not provable from $\mathrm{ZF} + \mathrm{PD}$: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).

    ↑

    the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).

    ↑

    the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)
  - fully understood (“**fine structure theory**”, Jensen and Silver, 1970s),

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
    - with ZF: (heuristically) complete description of finite sets;
    - with ZF+PD: (heuristically) complete description of finite and countable sets;
    - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

(think of $\mathbb{C}$ in the world of number fields of characteristic 0)

    - a typical candidate: Gödel's universe *L* of constructible sets (1938).

↑

the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)

    - fully understood (“**fine structure theory**”, Jensen and Silver, 1970s), but cannot be the reference universe because

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:
    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.
  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).
    ↑
    the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)
  - fully understood (“**fine structure theory**”, Jensen and Silver, 1970s),
    but cannot be the reference universe because
    - incompatible with large cardinals: contradicts PD,

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).

    ↑ the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)
  - fully understood (“**fine structure theory**”, Jensen and Silver, 1970s), but cannot be the reference universe because
    - incompatible with large cardinals: contradicts PD,
    - implies pathologies: existence of a non-measurable projective subset of $\mathbb{R}$...

- Fact: CH and ¬CH not provable from ZF + PD: description not yet complete...
  - with ZF: (heuristically) complete description of finite sets;
  - with ZF+PD: (heuristically) complete description of finite and countable sets;
  - with ZF+PD+??: (heuristically) complete description of sets up to cardinal $\aleph_1$:

    ... which will necessarily entail a solution to CH.

- Currently most promising approach: identify one canonical reference universe.

  (think of $\mathbb{C}$ in the world of number fields of characteristic 0)
  - a typical candidate: Gödel's universe L of constructible sets (1938).

    ↑

    the minimal universe: only definable sets (think of the prime field $\mathbb{Q}$)
  - fully understood ("**fine structure theory**", Jensen and Silver, 1970s), but cannot be the reference universe because
    - incompatible with large cardinals: contradicts PD,
    - implies pathologies: existence of a non-measurable projective subset of $\mathbb{R}$...

- Question: Can one find an L-like universe compatible with large cardinals?

- The inner model program (in the world of fields: constructing algebraic closure...)

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-**L *is true.*

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate**-L *is true.*

$\uparrow$

the L-like universe for one supercompact

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-L** *is true.*

↑
the L-like universe for one supercompact

- means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...)

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-L** *is true.*

↑
the L-like universe for one supercompact

- means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...) and useful (= provides a description with no pathologies)

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-L** *is true.*

↑
the L-like universe for one supercompact

  - means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...) and useful (= provides a description with no pathologies)

- Proposition: ZF + PD+V=ultimate-L *implies* GCH.

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-L** *is true.*

$\uparrow$

the L-like universe for one supercompact

- means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...) and useful (= provides a description with no pathologies)

- Proposition: ZF + PD+V=ultimate-L *implies* GCH.

  - If ZF + PD+V=ultimate-L becomes accepted as the base of Set Theory,

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-**$\boldsymbol{L}$ *is true.*

↑

the L-like universe for one supercompact

- means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...) and useful (= provides a description with no pathologies)

- Proposition: ZF + PD+V=ultimate-L *implies* GCH.

- If ZF + PD+V=ultimate-L becomes accepted as the base of Set Theory,
  then the Continuum Problem will have been solved.

- The inner model program (in the world of fields: constructing algebraic closure...)
  - universe $L[U]$ (Kunen, 1971): compatible with one measurable cardinal;
  - universe $L[E]$ (Mitchell–Steel, 1980-90s): compatible with PD;
  - but: how could this be completed with an endless hierarchy of large cardinals?

- Theorem (Woodin, 2006): *There exists an explicit level (one* **supercompact** *cardinal) such that, if an L-like universe is compatible with large cardinals up to that level, it is automatically compatible with all large cardinals.*

- Conjecture (Woodin, 2010): ZF + PD+V=**ultimate-L** *is true.*

↑
the L-like universe for one supercompact

- means proving that V=ultimate-L is both natural (an aesthetic judgment based on cumulated experience...) and useful (= provides a description with no pathologies)

- Proposition: ZF + PD+V=ultimate-L *implies* GCH.

- If ZF + PD+V=ultimate-L becomes accepted as the base of Set Theory, then the Continuum Problem will have been solved.

III. An application of a new type: Laver tables

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$:

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1				
2				
3				
4				

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1				
2				
3				
4				

- Start with $+1 \bmod 4$ in the first column,

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2				
3				
4				

- Start with $+1 \bmod 4$ in the first column,

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3				
4				

- Start with $+1 \bmod 4$ in the first column,

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4				

- Start with $+1 \bmod 4$ in the first column,

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - ▶ Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - ▶ Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - ▶ (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$4 * 2 =$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1)$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \qquad \text{(LD)}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1)$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - ▶ Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - ▶ Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - ▶ (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1			

- Start with $+1 \bmod 4$ in the first column, and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned} &4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\ &4 * 3 = 4 * (2 * 1) \end{aligned}$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1)$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2		

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	

- Start with $+1 \bmod 4$ in the first column, and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$
$$4 * 4$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 &= 4 * (3 * 1)
\end{aligned}$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned} 4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\ 4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\ 4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) \end{aligned}$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$
$$4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$
$$4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4			
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$
$$4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,$$
$$3 * 2 = 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4	4		
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 &= 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4,\ldots
\end{aligned}$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \qquad \text{(LD)}$$

  - Classical example 1: E module and $x * y := (1-\lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.

    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4	4	4	
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,

  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 &= 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots
\end{aligned}$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3			
3	4	4	4	4
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 &= 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots
\end{aligned}$$

A Laver table

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \tag{LD}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2			
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$\begin{aligned}
4 * 2 &= 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,\\
4 * 3 &= 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,\\
4 * 4 &= 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,\\
3 * 2 &= 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots
\end{aligned}$$

- The (left) selfdistributivity law:

$$x * (y * z) = (x * y) * (x * z). \qquad \text{(LD)}$$

  - Classical example 1: E module and $x * y := (1 - \lambda)x + \lambda y$;
  - Classical example 2: G group and $x * y := xyx^{-1}$.
    NB: all idempotent ($x * x = x$), hence no nontrivial monogenerated structure
  - (Brieskorn,...) Algebraic counterpart of Reidemeister move III...

- A binary operation on $\{1, 2, 3, 4\}$: the four element **Laver table**

$*$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

- Start with $+1 \bmod 4$ in the first column,
  and complete so as to obey the rule $x * (y * 1) = (x * y) * (x * 1)$:

$$4 * 2 = 4 * (1 * 1) = (4 * 1) * (4 * 1) = 1 * 1 = 2,$$
$$4 * 3 = 4 * (2 * 1) = (4 * 2) * (4 * 1) = 2 * 1 = 3,$$
$$4 * 4 = 4 * (3 * 1) = (4 * 3) * (4 * 1) = 3 * 1 = 4,$$
$$3 * 2 = 3 * (1 * 1) = (3 * 1) * (3 * 1) = 4 * 4 = 4, \ldots$$

- The same construction works for every size:

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on $\{1, ..., N\}$ satisfying*

$$x * 1 = x + 1 \bmod N \text{ and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on $\{1, ..., N\}$ satisfying*

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) *The operation thus obtained obeys the LD-law if and only if N is a power of 2.*

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on* $\{1, \ldots, N\}$ *satisfying*

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) *The operation thus obtained obeys the LD-law if and only if N is a power of 2.*

  - ▶ A_n := the **Laver table** with 2^n elements.

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on $\{1, \ldots, N\}$ satisfying*

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) *The operation thus obtained obeys the LD-law if and only if N is a power of* 2.

  - A_n := the **Laver table** with 2^n elements.

- For $n \geqslant 1$, one has $1 * 1 = 2 \neq 1$ in A_n: not idempotent.

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on $\{1, ..., N\}$ satisfying*

$$x * 1 = x + 1 \bmod N \ \text{ and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) *The operation thus obtained obeys the LD-law if and only if N is a power of* 2.

  - ▶ $A_n :=$ the **Laver table** with 2^n elements.

- For $n \geqslant 1$, one has $1 * 1 = 2 \neq 1$ in A_n: not idempotent.
  - ▶ quite different from group conjugacy and other classical LD-structures

- The same construction works for every size:

- Proposition (Laver): (i) *For every N, there exists a unique binary operation $*$ on $\{1, \ldots, N\}$ satisfying*

$$\begin{gathered} x * 1 = x + 1 \bmod N \ \text{ and} \\ x * (y * 1) = (x * y) * (x * 1). \end{gathered}$$

(ii) *The operation thus obtained obeys the LD-law if and only if N is a power of* 2.

  - ▸ $A_n :=$ the **Laver table** with 2^n elements.

- For $n \geqslant 1$, one has $1 * 1 = 2 \neq 1$ in A_n: not idempotent.
  - ▸ quite different from group conjugacy and other classical LD-structures
  - ▸ a counterpart of cyclic groups $\mathbb{Z}/n\mathbb{Z}$ in the selfdistributive world:

- The same construction works for every size:

- <u>Proposition</u> (Laver): (i) *For every* N*, there exists a unique binary operation* $*$ *on* $\{1, ..., N\}$ *satisfying*

$$x * 1 = x + 1 \bmod N \quad \text{and}$$
$$x * (y * 1) = (x * y) * (x * 1).$$

(ii) *The operation thus obtained obeys the LD-law if and only if* N *is a power of* 2*.*

  - A_n := the **Laver table** with 2^n elements.

- For $n \geqslant 1$, one has $1 * 1 = 2 \neq 1$ in A_n: not idempotent.
  - quite different from group conjugacy and other classical LD-structures
  - a counterpart of cyclic groups $\mathbb{Z}/n\mathbb{Z}$ in the selfdistributive world:
    A_n presented by $\langle 1 \mid 1_{[2^n]} = 1 \rangle_{LD}$, with $x_{[p]} = (...((x * x) * x)...) * x$, p terms.

$\mathbf{A_0}$	1
1	1

Laver tables: examples

$\mathbf{A_0}$	1
1	1

$\mathbf{A_1}$	1	2
1	2	2
2	1	2

Laver tables: examples

$\mathbf{A}_0$	1
1	1

$\mathbf{A}_1$	1	2
1	2	2
2	1	2

$\mathbf{A}_2$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

Laver tables: examples

$\mathbf{A}_0$	1
1	1

$\mathbf{A}_1$	1	2
1	2	2
2	1	2

$\mathbf{A}_2$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$\mathbf{A}_3$	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Laver tables: examples

$\mathbf{A}_0$	1
1	1

$\mathbf{A}_1$	1	2
1	2	2
2	1	2

$\mathbf{A}_2$	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

$\mathbf{A}_3$	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\mathbf{A}_4$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
4	5	6	7	8	13	14	15	16	5	6	7	8	13	14	15	16
5	6	8	14	16	6	8	14	16	6	8	14	16	6	8	14	16
6	7	8	15	16	7	8	15	16	7	8	15	16	7	8	15	16
7	8	16	8	16	8	16	8	16	8	16	8	16	8	16	8	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	10	12	14	16	10	12	14	16	10	12	14	16	10	12	14	16
10	11	12	15	16	11	12	15	16	11	12	15	16	11	12	15	16
11	12	16	12	16	12	16	12	16	12	16	12	16	12	16	12	16
12	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
13	14	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16
14	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2,

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the* p*th row of* A_n

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the* p*th row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^{\mathrm{n}}$ *to* 2^n.

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the* p*th row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^{\mathrm{n}}$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p+1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

► $\pi_3(\mathbf{8}) = 8$

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- Proposition (Laver): *For every* $p \leqslant 2^n$*, there exists a number* $\pi_n(p)$*, a power of* 2*, such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n*.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\boldsymbol{\pi_3(8)} = 8$

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p+1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_{\mathbf{3}}(\mathbf{8}) = 8$

• Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the* p*th row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

• Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\boldsymbol{\pi_3}(\mathbf{8}) = 8$

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\boldsymbol{\pi_3(8)} = 8$

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\boldsymbol{\pi_3(8)} = 8$

- A few values of the periods of 1 and 2:

n	
$\pi_n(1)$	
$\pi_n(2)$	

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0
$\pi_n(1)$	1
$\pi_n(2)$	–

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1
$\pi_n(1)$	1	1
$\pi_n(2)$	–	2

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2
$\pi_n(1)$	1	1	2
$\pi_n(2)$	–	2	2

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3
$\pi_n(1)$	1	1	2	4
$\pi_n(2)$	–	2	2	4

- **Proposition** (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4
$\pi_n(1)$	1	1	2	4	4
$\pi_n(2)$	–	2	2	4	4

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\boldsymbol{\pi_3(8)} = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5
$\pi_n(1)$	1	1	2	4	4	8
$\pi_n(2)$	–	2	2	4	4	8

- **Proposition** (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6
$\pi_n(1)$	1	1	2	4	4	8	8
$\pi_n(2)$	–	2	2	4	4	8	8

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7
$\pi_n(1)$	1	1	2	4	4	8	8	8
$\pi_n(2)$	–	2	2	4	4	8	8	16

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8
$\pi_n(1)$	1	1	2	4	4	8	8	8	8
$\pi_n(2)$	–	2	2	4	4	8	8	16	16

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▸ $\pi_3(1) = 4$
- ▸ $\pi_3(2) = 4$
- ▸ $\pi_3(3) = 2$
- ▸ $\pi_3(4) = 4$
- ▸ $\pi_3(5) = 2$
- ▸ $\pi_3(6) = 2$
- ▸ $\pi_3(7) = 1$
- ▸ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\boldsymbol{\pi_3(8)} = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	$\cdots$
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	$\cdots$

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

  - $\pi_3(1) = 4$
  - $\pi_3(2) = 4$
  - $\pi_3(3) = 2$
  - $\pi_3(4) = 4$
  - $\pi_3(5) = 2$
  - $\pi_3(6) = 2$
  - $\pi_3(7) = 1$
  - $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	$\cdots$
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	$\cdots$

  - Question 1: Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?

- <u>Proposition</u> (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^n$ to 2^n.*

- <u>Example</u>:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

  - $\pi_3(1) = 4$
  - $\pi_3(2) = 4$
  - $\pi_3(3) = 2$
  - $\pi_3(4) = 4$
  - $\pi_3(5) = 2$
  - $\pi_3(6) = 2$
  - $\pi_3(7) = 1$
  - $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	$\cdots$
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	$\cdots$

  - <u>Question 1</u>: Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?
  - <u>Question 2</u>: Does $\pi_n(1)$ tend to ∞ with n?

- Proposition (Laver): *For every* $p \leqslant 2^n$, *there exists a number* $\pi_n(p)$, *a power of* 2, *such that the pth row of* A_n *is the periodic repetition of* $\pi_n(p)$ *values increasing from* $p{+}1 \bmod 2^n$ *to* 2^n.

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- ▶ $\pi_3(1) = 4$
- ▶ $\pi_3(2) = 4$
- ▶ $\pi_3(3) = 2$
- ▶ $\pi_3(4) = 4$
- ▶ $\pi_3(5) = 2$
- ▶ $\pi_3(6) = 2$
- ▶ $\pi_3(7) = 1$
- ▶ $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	⋯
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	⋯
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	⋯

  - ▶ Question 1: Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?
  - ▶ Question 2: Does $\pi_n(1)$ tend to ∞ with n? Does it reach 32?

- Proposition (Laver): *For every $p \leqslant 2^n$, there exists a number $\pi_n(p)$, a power of 2, such that the pth row of A_n is the periodic repetition of $\pi_n(p)$ values increasing from $p{+}1 \bmod 2^{\mathrm{n}}$ to 2^n.*

- Example:

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

- $\pi_3(1) = 4$
- $\pi_3(2) = 4$
- $\pi_3(3) = 2$
- $\pi_3(4) = 4$
- $\pi_3(5) = 2$
- $\pi_3(6) = 2$
- $\pi_3(7) = 1$
- $\pi_3(\mathbf{8}) = 8$

- A few values of the periods of 1 and 2:

n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\pi_n(1)$	1	1	2	4	4	8	8	8	8	16	16	16	$\cdots$
$\pi_n(2)$	–	2	2	4	4	8	8	16	16	16	16	16	$\cdots$

- Question 1: Does $\pi_n(2) \geqslant \pi_n(1)$ always hold?
- Question 2: Does $\pi_n(1)$ tend to ∞ with n? Does it reach 32?

- Theorem (Laver, 1995):

  the answer to the above questions is positive.

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank**

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$.

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)
- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)
- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R,

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$,

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)
- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$,

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$,

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell){=}i(j)(i(k))$, *i.e.*, $i(j(k)){=}i(j)(i(k))$:

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)
- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, *i.e.*, $i(j(k)) = i(j)(i(k))$: LD-law.

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, *i.e.*, $i(j(k)) = i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*
  (i) The set Iter(j) *of* **iterates** *of j* (*i.e.*, j, $j(j)$, $j(j)(j)$...) *obeys the LD-law.*

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, i.e., $i(j(k)) = i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*
  (i) The set Iter(j) *of* **iterates** *of j* (i.e., j, $j(j)$, $j(j)(j)$...) *obeys the LD-law.*
  (ii) For every n, there exists a compatible equivalence relation on Iter(j) *with 2^n classes and the first column of its table is a cycle,*

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell)=i(j)(i(k))$, *i.e.*, $i(j(k))=i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*
  (i) The set Iter(j) *of* **iterates** *of j* (*i.e.*,j, $j(j)$, $j(j)(j)$...) *obeys the LD-law.*
  (ii) For every n, there exists a compatible equivalence relation on Iter(j) *with 2^n classes and the first column of its table is a cycle, hence the quotient is A_n.*

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,

    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, i.e., $i(j(k)) = i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*

  (i) The set Iter(j) *of* **iterates** *of j* (i.e., j, $j(j)$, $j(j)(j)$...) *obeys the LD-law.*

  (ii) For every n, there exists a compatible equivalence relation on Iter(j) *with 2^n classes and the first column of its table is a cycle, hence the quotient is A_n.*

  (iii) For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,
    hence $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, i.e., $i(j(k)) = i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*

  (i) The set Iter(j) *of* **iterates** *of j (i.e., j, $j(j)$, $j(j)(j)$...) obeys the LD-law.*

  (ii) For every n, there exists a compatible equivalence relation on Iter(j) *with 2^n classes and the first column of its table is a cycle, hence the quotient is A_n.*

  (iii) For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit}(j_{[2^m]})$ *to* $\mathrm{crit}(\mathrm{j}_{[2^n]})$.

- Theorem (Laver, 1995): *If there exists a selfsimilar set, then the answer to the above questions is positive.*

- Definition: A **rank** is a set R such that $f : R \to R$ implies $f \in R$. (this exists...)

- Assume that X is selfsimilar (i.e., $\exists$ self-embedding of X):
  - then there exists a selfsimilar rank, say R;
  - if i, j are self-embeddings of R, then $i : R \to R$ and $j \in R$, hence we can **apply** i to j;
  - "being a self-embedding" is definable from $\in$, hence $i(j)$ is a self-embedding;
  - "being the image of" is definable from $\in$,

    hence $\ell = j(k)$ implies $i(\ell)=i(j)(i(k))$, i.e., $i(j(k))=i(j)(i(k))$: LD-law.

- Proposition (Laver): *Assume j is a self-embedding of a rank R.*

  (i) The set Iter(j) *of* **iterates** *of j* (i.e., j, $j(j)$, $j(j)(j)$...) *obeys the LD-law.*

  (ii) For every n, there exists a compatible equivalence relation on Iter(j) *with 2^n classes and the first column of its table is a cycle, hence the quotient is A_n.*

  (iii) For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit}(j_{[2^m]})$ *to* $\mathrm{crit}(j_{[2^n]})$.

↑

the first ordinal moved by...

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  - Proof:

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*
  - Proof: There exists β satisfying $j(\beta) > \alpha$,

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  - Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β,

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \tag{$*$}$$

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

▸ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \tag{$*$}$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

▸ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall\gamma < \beta \; (j(\gamma) \leqslant \alpha). \tag{$*$}$$

Applying j to $(*)$ gives

$$\forall\gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$

Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall\gamma < \beta\ (j(\gamma) \leqslant \alpha). \tag{$*$}$$

Applying j to $(*)$ gives

$$\forall\gamma < j(\beta)\ (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$

Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  - Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
  $$\forall\gamma < \beta\ (j(\gamma) \leqslant \alpha). \qquad (*)$$
  Applying j to $(*)$ gives
  $$\forall\gamma < j(\beta)\ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$
  Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  ▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
  $$\forall\gamma < \beta \; (j(\gamma) \leqslant \alpha). \tag{$*$}$$
  Applying j to $(*)$ gives
  $$\forall\gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$
  Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Alternative proofs without the large cardinal assumption?

- <u>Lemma</u>: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  ▸ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

  $$\forall\gamma < \beta\ (j(\gamma) \leqslant \alpha). \tag{$*$}$$

  Applying j to $(*)$ gives

  $$\forall\gamma < j(\beta)\ (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$

  Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- <u>Proposition</u> (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Alternative proofs without the large cardinal assumption? Not yet...

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  ▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
  $$\forall\gamma < \beta\ (j(\gamma) \leqslant \alpha). \tag{$*$}$$
  Applying j to $(*)$ gives
  $$\forall\gamma < j(\beta)\ (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$
  Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Alternative proofs without the large cardinal assumption? Not yet...
  - ▶ partial results by Drápal...

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

  ▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

  $$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

  Applying j to $(*)$ gives

  $$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

  Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. ☐

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Alternative proofs without the large cardinal assumption? Not yet...
  - ▶ partial results by Drápal... but no complete proof so far:

- Lemma: *If j is a self-embedding, then $j(j)(\alpha) \leqslant j(\alpha)$ holds for every ordinal α.*

   - Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \tag{$*$}$$
Applying j to $(*)$ gives
$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \tag{$**$}$$
Taking $\gamma = \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a selfsimilar set, $\pi_n(2) \geqslant \pi_n(1)$ holds for every n.*

- Similar (more difficult) proof for Question 1 (period of 1 in A_n tends to ∞).

- Alternative proofs without the large cardinal assumption? Not yet...
   - partial results by Drápal... but no complete proof so far:
   - a strange situation: why a connection between finite tables and large cardinals?

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.

(nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes;

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: "New Math" was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
(nothing less, nothing more: "New Math" was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy:

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: "New Math" was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof;

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities. (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass

them to the mathematician for a formal proof; Here: using a logical intuition
(existence of a selfsimiliar set),

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof; Here: using a logical intuition (existence of a selfsimiliar set), guess statements (periods in Laver tables tend to ∞),

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: "New Math" was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof; Here: using a logical intuition (existence of a selfsimiliar set), guess statements (periods in Laver tables tend to ∞), then pass them to the mathematician for a formal proof.

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
  (nothing less, nothing more: “New Math” was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof; Here: using a logical intuition (existence of a selfsimiliar set), guess statements (periods in Laver tables tend to ∞), then pass them to the mathematician for a formal proof.
  - No need to believe in the existence of large cardinals to use them...

- Set Theory is a theory of infinity: its aim is to explore the various possible infinities.
(nothing less, nothing more: "New Math" was a misunderstanding...)

- History continues:
  - A coherent theory beyond ZF is possible;
  - There is a consensus about enriching ZF into ZF+PD;
  - The next step should include a solution of the Continuum Problem.

- A last question: Are the properties of Laver tables an application of Set Theory?
  - **So far**, yes; **later**, formally no if one finds alternative proofs without Set Theory.
  - But, **in any case**, it is Set Theory that made the properties first accessible...
  - An analogy: In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof; Here: using a logical intuition (existence of a selfsimiliar set), guess statements (periods in Laver tables tend to ∞), then pass them to the mathematician for a formal proof.
  - No need to believe in the existence of large cardinals to use them...

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*, Proceedings ICM Hyderabad 2010, pp. 504–528

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*,
  Proceedings ICM Hyderabad 2010, pp. 504–528

- R. Laver, *On the algebra of elementary embeddings of a rank into itself*,
  Adv. in Math. 110 (1995) 334–346

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*, Proceedings ICM Hyderabad 2010, pp. 504–528
- R. Laver, *On the algebra of elementary embeddings of a rank into itself*, Adv. in Math. 110 (1995) 334–346
- P. Dehornoy, *Laver's results and low-dimensional topology*, Arch. Math. Logic, 55 (2016) 49–83.

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*,
  Proceedings ICM Hyderabad 2010, pp. 504–528
- R. Laver, *On the algebra of elementary embeddings of a rank into itself*,
  Adv. in Math. 110 (1995) 334–346
- P. Dehornoy, *Laver's results and low-dimensional topology*,
  Arch. Math. Logic, 55 (2016) 49–83.
- P. Dehornoy & V. Lebed, *Two- and three-cocycles for Laver tables*,
  J. Knot Theory and Ramifications, 23-4 (2014) 1450017

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*, Proceedings ICM Hyderabad 2010, pp. 504–528

- R. Laver, *On the algebra of elementary embeddings of a rank into itself*, Adv. in Math. 110 (1995) 334–346

- P. Dehornoy, *Laver's results and low-dimensional topology*, Arch. Math. Logic, 55 (2016) 49–83.

- P. Dehornoy & V. Lebed, *Two- and three-cocycles for Laver tables*, J. Knot Theory and Ramifications, 23-4 (2014) 1450017

www.math.unicaen.fr/∼dehornoy

- W. Hugh Woodin, *Strong axioms of infinity and the search for V*,
  Proceedings ICM Hyderabad 2010, pp. 504–528

- R. Laver, *On the algebra of elementary embeddings of a rank into itself*,
  Adv. in Math. 110 (1995) 334–346

- P. Dehornoy, *Laver's results and low-dimensional topology*,
  Arch. Math. Logic, 55 (2016) 49–83.

- P. Dehornoy & V. Lebed, *Two- and three-cocycles for Laver tables*,
  J. Knot Theory and Ramifications, 23-4 (2014) 1450017

www.math.unicaen.fr/∼dehornoy

... Muito obrigado e da próxima vez !