Garside germs for YBE structure groups, and an extension of Ore's theorem

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  - W. Rump's formalism of cycle sets for investigating YBE structure groups: revisit the Garside and the I-structures, and introduce a finite Coxeter-like quotient,
  - a new approach to the word problem of Artin-Tits groups, based on an extension of Ore's theorem from fractions to multifractions.

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• Definition: A set-theoretic solution of YBE is a pair (S, r) where S is a set and r is a bijection from $S \times S$ to itself satisfying

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- <u>Theorem</u> (Rump, 2005): (i) If (S, r) is an involutive nondegenerate solution, then $(S, *)$ is a bijective RC-quasigroup, where $s * t :=$ the unique r s.t. $r_1(s, r) = t$.

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(ii) Conversely, is $(S, *)$ is a bijective RC-quasigroup, then (S, r) is an involutive nondegenerate solution, where $r(a, b) :=$ the unique pair (a', b') s.t. $a * a' = b$ and $a' * a = b'$.

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Similarly, for $n \geqslant 1$, let (where $\cdot$ is another binary operation):

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  - Think of the Ω_n as (counterparts of) iterated sums in the RC-world, and of the Π_n as (counterparts of) iterated products.

- Lemma: *If $(S, *)$ is a cycle set, then, for every π in $\mathfrak{S}_{n-1}$, one has*
$$\Omega_n(s_{\pi(1)}, ..., s_{\pi(n-1)}, s_n) = \Omega_n(s_1, ..., s_n).$$
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- Lemma: If $(S, *)$ is a bijective RC-quasigroup, there exists $\tilde{*}$, unique, s.t.
$$(s, t) \mapsto (s \mathbin{\tilde{*}} t, t \mathbin{\tilde{*}} s) \text{ is the inverse of } (s, t) \mapsto (s * t, t * s).$$
Then $(S, \tilde{*})$ is a bijective LC-quasigroup and, for $\widetilde{s}_i := \Omega_n(s_1, ..., \widehat{s}_i, , ..., s_n, s_i)$, one has
$$\Omega_i(s_{\pi(1)}, ..., s_{\pi(i)}) = \widetilde{\Omega}_{n+1-i}(\widetilde{s}_{\pi(i)}, ..., \widetilde{s}_{\pi(n)}),$$
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- Claim: One can (easily) develop an "RC-calculus".

- <u>Definition</u>: For $n \geqslant 1$, define $\Omega_1(x_1) := x_1$ and

$$\Omega_n(x_1, \dots, x_n) := \Omega_{n-1}(x_1, \dots, x_{n-1}) * \Omega_{n-1}(x_1, \dots, x_{n-2}, x_n).$$

Similarly, for $n \geqslant 1$, let (where $\cdot$ is another binary operation):

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  - "Inversion formulas"; etc. etc.

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    $\mathrm{Catalan}_n$ simples ($\cong$ noncrossing partitions), the maximal one δ_n of length $n-1$.

- Why do we care about Garside structures?
  - The word problem is solvable (in quadratic time).
  - There is a canonical normal form for the elements ("greedy normal form").
  - There is a (bi)-automatic structure.
  - The (co)homology is efficiently computable.
  - There is no torsion.

The whole structure is encoded in the (finite) family of simple elements.

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- Once again, the proof seems easier and more explicit when stated in terms of $*$ and RC.

Plan:

- Structure groups of set-theoretic solutions of YBE
  - ▶ 1. Rump's RC-calculus
    - Solutions of YBE vs. biracks vs. cycle sets
    - Revisiting the Garside structure using RC-calculus
    - Revisiting the I-structure using RC-calculus
  - ▶ 2. A new application: Garside germs
    - The braid germ
    - The YBE germ
- A new approach to the word problem of Artin-Tits groups
  - ▶ 3. Multifraction reduction, an extension of Ore's theorem
    - Ore's classical theorem
    - Extending free reduction: (i) division, (ii) reduction
    - The case of Artin-Tits groups: theorems and conjectures

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• Definition: A group W with generating family Σ is a germ for a monoid M if there exists a projection $\pi : M \to W$ and a section $\sigma : W \hookrightarrow M$ of π s.t. M is generated by $\mathrm{Im}(\sigma)$ and (the counterpart of) $(*)$ holds.

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$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} & |i-j| = 1 \end{array} \right\rangle.$$

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$(*) \qquad \sigma(f)\sigma(g) = \sigma(fg)$ *holds in* B_n^+ *iff* $\ell_\Sigma(f) + \ell_\Sigma(g) = \ell_\Sigma(fg)$ *holds in* $\mathfrak{S}_n$,

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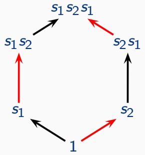

Cayley graph of $\mathfrak{S}_3$
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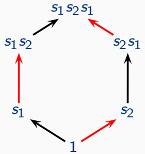

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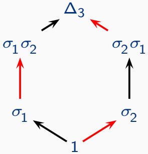

Hasse diagram of left division
for simple elements of B_3^+

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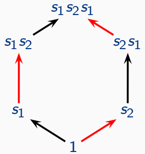

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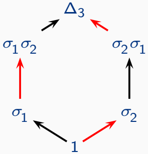

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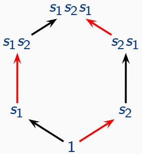

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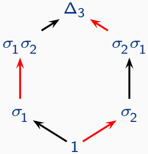

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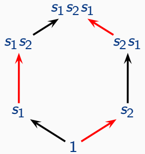

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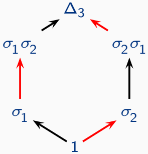

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The braid germ (cont'd)

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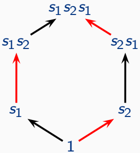

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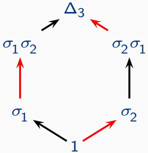

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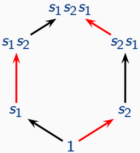

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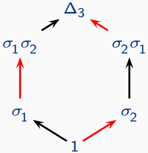

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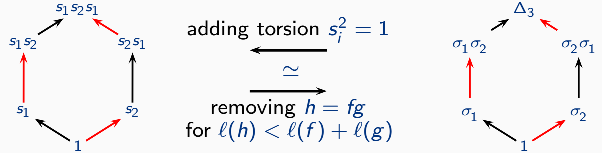

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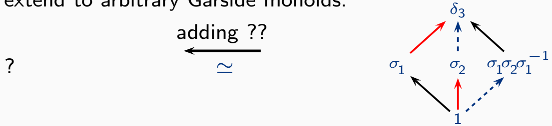

Hasse diagram of left division
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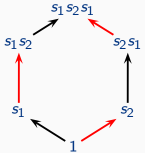

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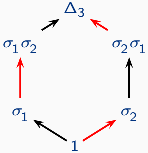

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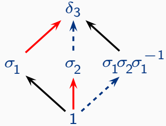

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Then $\overline{G}$ is $G/\equiv$ where $g \equiv g'$ means $\forall s{\in}S\ (\#_s(\nu^{-1}(g)) = \#_s(\nu^{-1}(g')) \bmod d)$. □

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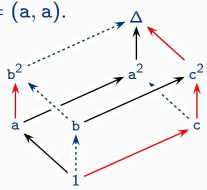

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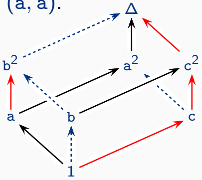

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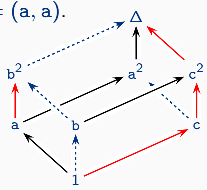

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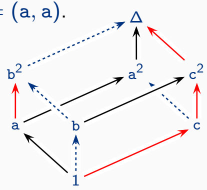

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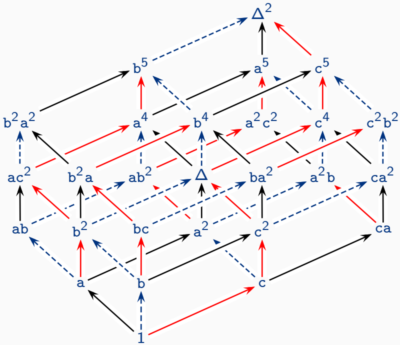

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- P. Dehornoy, with F. Digne, E. Godelle, D. Krammer, J. Michel, Chapter XIII of: Foundations of Garside Theory, EMS Tracts in Mathematics, vol. 22 (2015)

Plan:

- Structure groups of set-theoretic solutions of YBE
  - ▶ 1. Rump's RC-calculus
    - Solutions of YBE vs. biracks vs. cycle sets
    - Revisiting the Garside structure using RC-calculus
    - Revisiting the I-structure using RC-calculus
  - ▶ 2. A new application: Garside germs
    - The braid germ
    - The YBE germ
- A new approach to the word problem of Artin-Tits groups
  - ▶ 3. Multifraction reduction, an extension of Ore's theorem
    - Ore's classical theorem
    - Extending free reduction: (i) division, (ii) reduction
    - The case of Artin-Tits groups: theorems and conjectures

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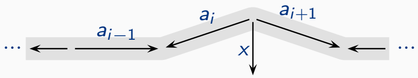

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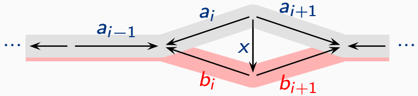

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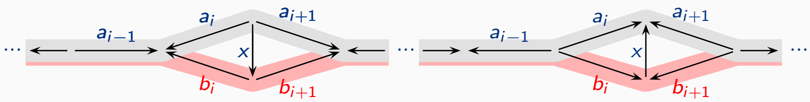

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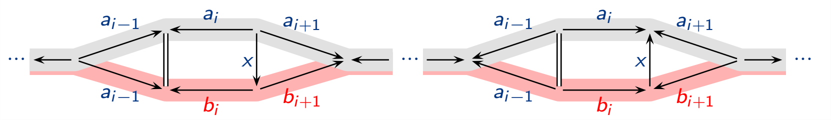

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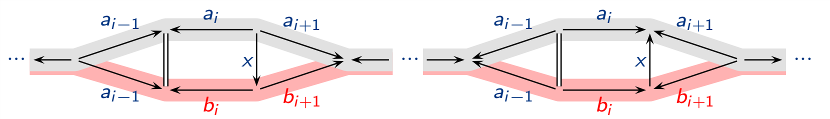

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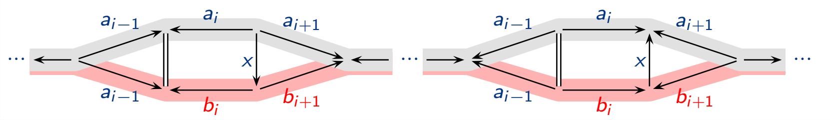

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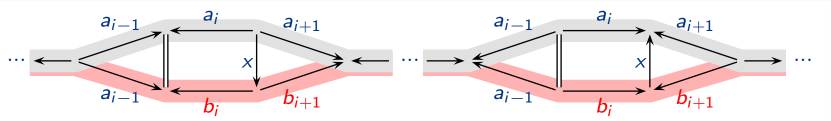

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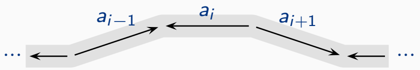

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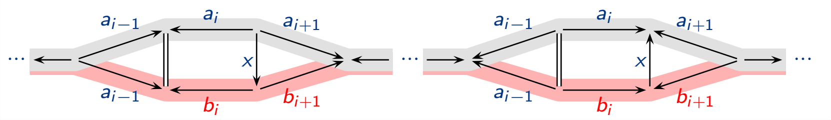

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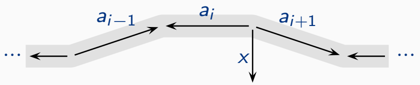

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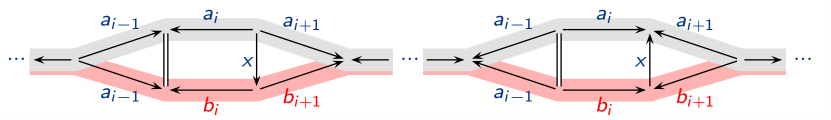

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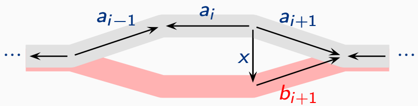

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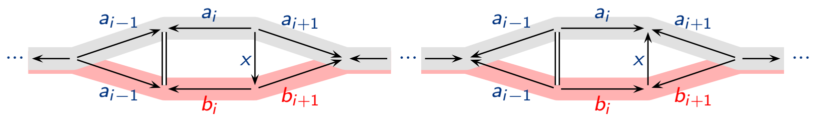

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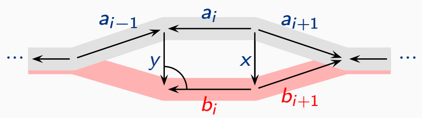

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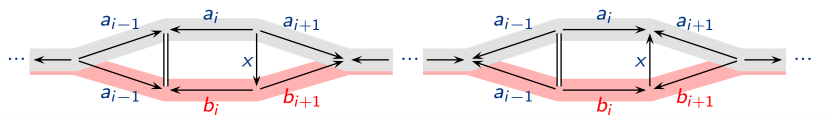

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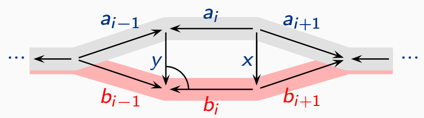

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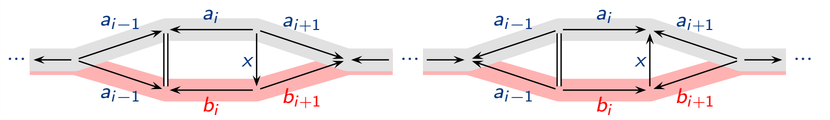

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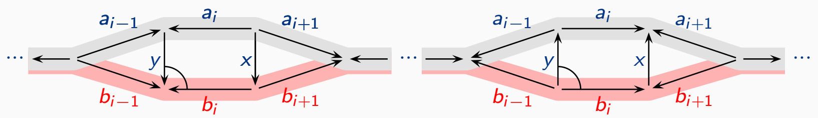

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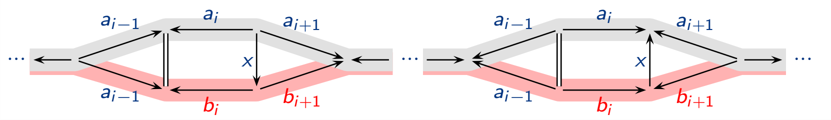

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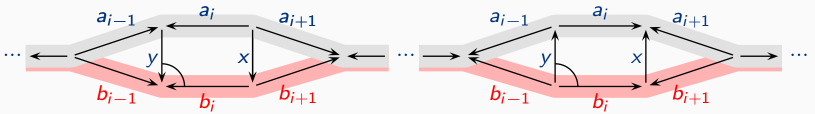

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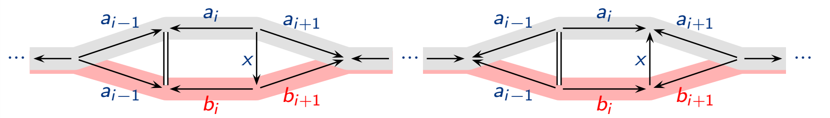

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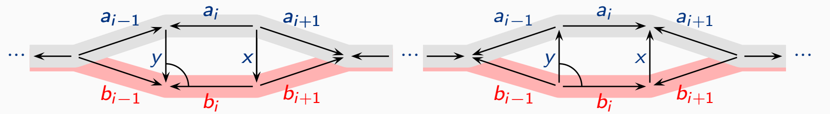

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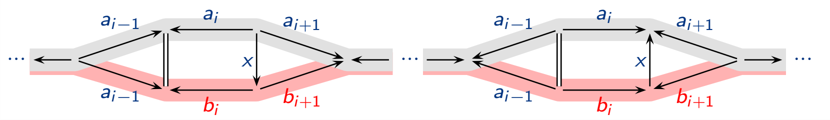

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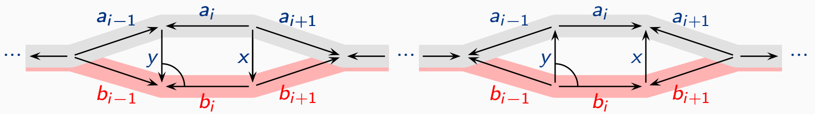

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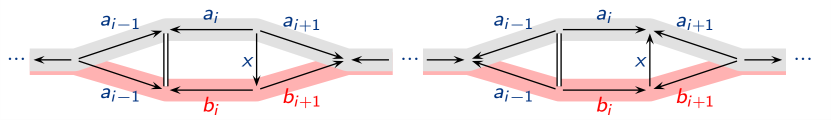

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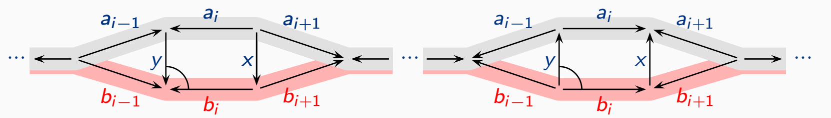

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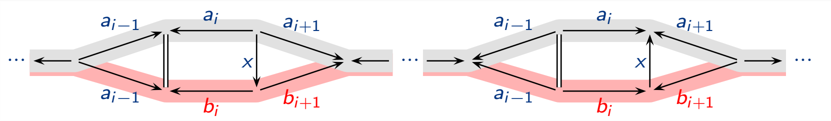

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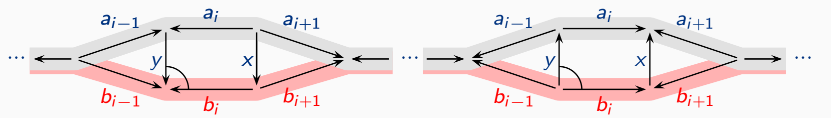

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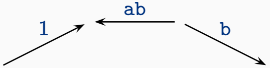
  - Proof: We walk in the Cayley graph, replacing one path with an equivalent one. $\square$

- Example: $M = B_3^+$ with $\mathtt{1}/\mathtt{ab}/\mathtt{b}$:



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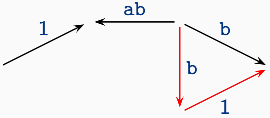

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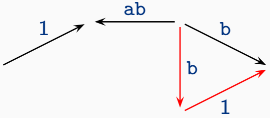

`b` and `ab` admit a common multiple

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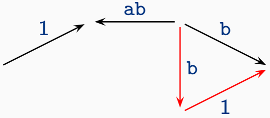

`b` and `ab` admit a common multiple
  - we can push `b` through `ab`:

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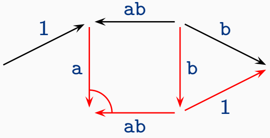
- Example: $M = B_3^+$ with 1/ab/b:



b and ab admit a common multiple
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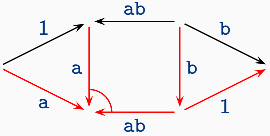

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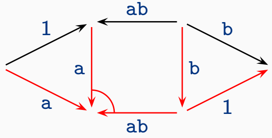

b and ab admit a common multiple
- we can push b through ab:
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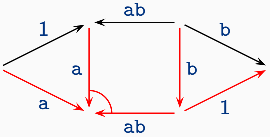

$\mathtt{b}$ and $\mathtt{ab}$ admit a common multiple
- we can push $\mathtt{b}$ through $\mathtt{ab}$:
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and now

$$\begin{tikzcd} & \mathtt{a}/\mathtt{aba}/\mathtt{b} \arrow[dl, Rightarrow, "D_{1,\mathtt{a}}"'] \arrow[dr, Rightarrow, "D_{2,\mathtt{b}}"] & \\ \mathtt{1}/\mathtt{ab}/\mathtt{b} & & \mathtt{a}/\mathtt{ab}/\mathtt{1} \end{tikzcd}$$

Reduction (cont'd)

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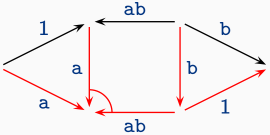

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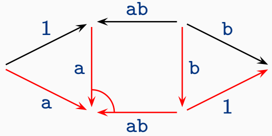

$\mathtt{b}$ and $\mathtt{ab}$ admit a common multiple
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- <u>Fact</u>: $\underline{b} = \underline{a} \bullet R_{i,x}$ implies that $\underline{a}$ and $\underline{b}$ represent the same element in $\mathcal{U}(M)$.
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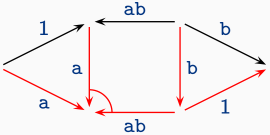

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- In this way: for every gcd-monoid M, a rewrite system $\mathcal{R}_M$ ("reduction").

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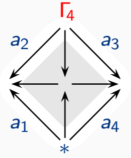

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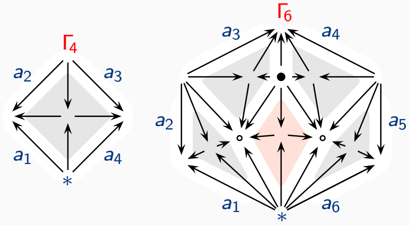

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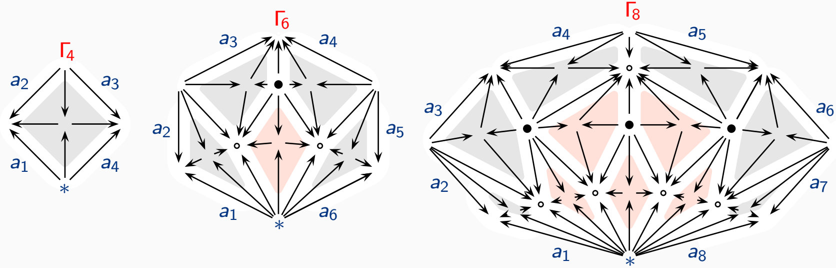

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- References:

  - P. Dehornoy, *Multifraction reduction I: The 3-Ore case and Artin-Tits groups of type FC*, J. Comb. Algebra 1 (2017) 185–228
  - P. Dehornoy, *Multifraction reduction II: Conjectures for Artin-Tits groups* J. Comb. Algebra, to appear, arXiv:1606.08995
  - P. Dehornoy & F. Wehrung, *Multifraction reduction III: The case of interval monoids* J. Comb. Algebra, to appear, arXiv:1606.09018
  - P. Dehornoy, D. Holt, & S. Rees, *Multifraction reduction IV: Padding and Artin-Tits groups of sufficiently large type*, arXiv:1701.06413

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