Self-distributivity, braids, and set theory

Self-distributivity, braids, and set theory

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

Self-distributivity, braids, and set theory

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

Self-distributive systems and quandle (co)homology theory
in algebra and low-dimensional topology, Pusan, June 2017

Self-distributivity, braids, and set theory

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

Self-distributive systems and quandle (co)homology theory
in algebra and low-dimensional topology, Pusan, June 2017

Self-distributivity, braids, and set theory

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

Self-distributive systems and quandle (co)homology theory
in algebra and low-dimensional topology, Pusan, June 2017

- Many things are known about shelves (SD-structures that need not be racks).

Self-distributivity, braids, and set theory

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen

Self-distributive systems and quandle (co)homology theory
in algebra and low-dimensional topology, Pusan, June 2017

- Many things are known about shelves (SD-structures that need not be racks).
- Here special emphasis on the connection with braids and with set theory.

Plan:

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf

Plan:

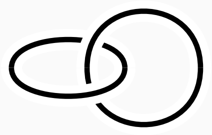
- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

Plan:

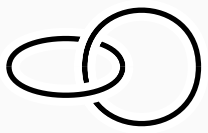
- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

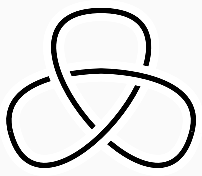
- Planar diagrams:

- Planar diagrams:

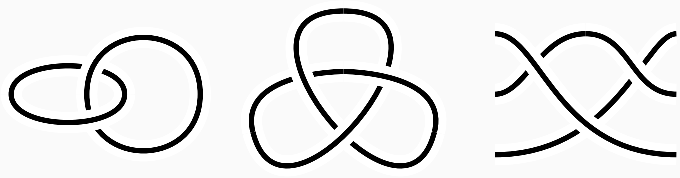


- Planar diagrams:

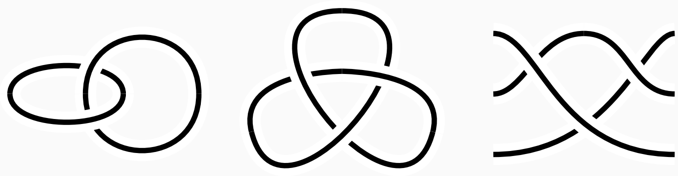




- Planar diagrams:

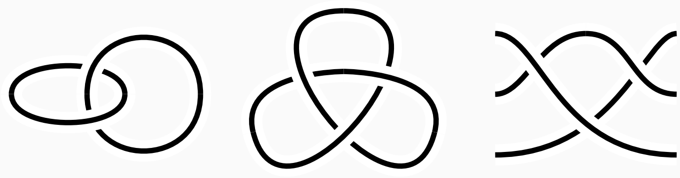


- Planar diagrams:



  - projections of curves embedded in $\mathbb{R}^3$

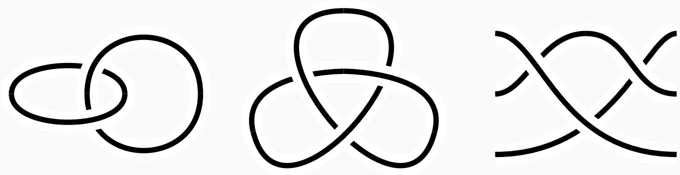
- Planar diagrams:



  - projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two diagrams are (projections of) isotopic figures

- Planar diagrams:



▶ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two diagrams are (projections of) isotopic figures

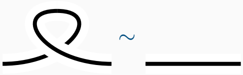
▶ find isotopy invariants.

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I : 

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I : 

- type II : 

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

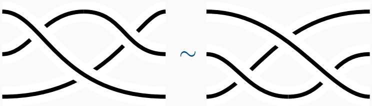
- type III :

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

- type III :



- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

and

.

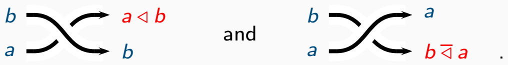
- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

and

.

- Action of Reidemeister moves on colors:

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$, and color the strands in diagrams obeying the rules:



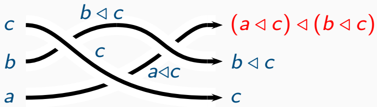
- Action of Reidemeister moves on colors:

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

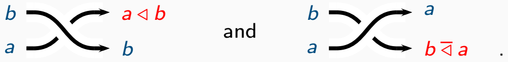
and

.

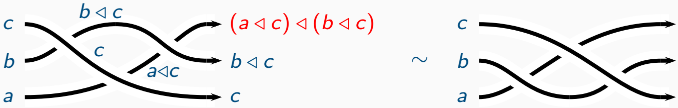
- Action of Reidemeister moves on colors:



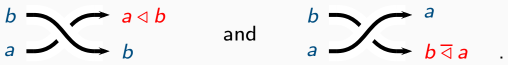
- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$, and color the strands in diagrams obeying the rules:



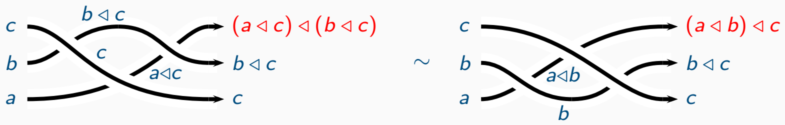
- Action of Reidemeister moves on colors:



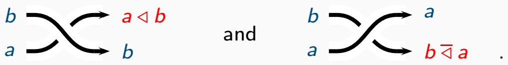
- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:



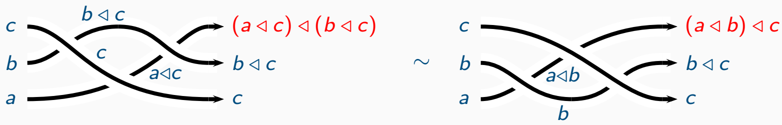
- Action of Reidemeister moves on colors:



- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$, and color the strands in diagrams obeying the rules:



- Action of Reidemeister moves on colors:



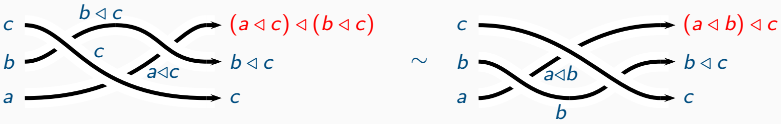
▸ Hence: S-colorings invariant under Reidemeister move III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

and

.

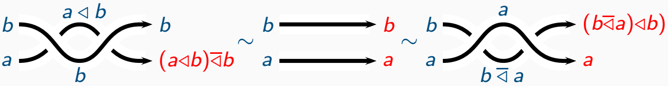
- Action of Reidemeister moves on colors:



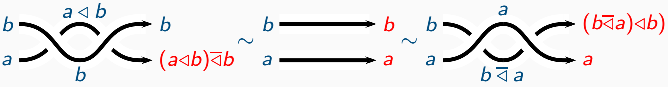
  - ▸ Hence: S-colorings invariant under Reidemeister move III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf

- <u>Proposition</u>: *Whenever $(S, \triangleleft)$ is a shelf, diagram coloring provides a well defined action of the braid monoid B_n^+ on S^n for every n.*

- Idem for Reidemeister move II:

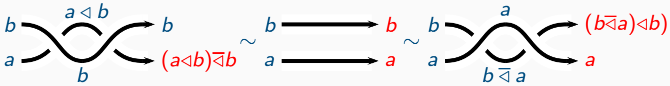


- Idem for Reidemeister move II:



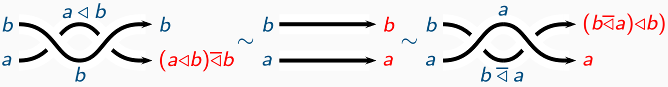
- Lemma: *There exists* $\overline{\triangleleft}$ *satisfying* $(x \triangleleft y) \overline{\triangleleft} y = x$ *and* $(x \overline{\triangleleft} y) \triangleleft y = x$ *iff the right translations of* $(S, \triangleleft)$ *are bijections.*

- Idem for Reidemeister move II:



- Lemma: *There exists* $\bar{\triangleleft}$ *satisfying* $(x \triangleleft y) \,\bar{\triangleleft}\, y = x$ *and* $(x \,\bar{\triangleleft}\, y) \triangleleft y = x$ *iff the right translations of* $(S, \triangleleft)$ *are bijections.*
  - Hence: S-colorings invariant under Reidemeister moves II+III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf with bijective right translations

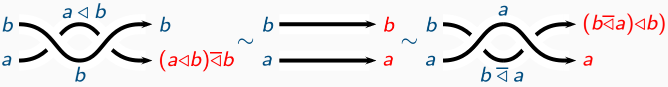
- Idem for Reidemeister move II:



- Lemma: *There exists* $\overline{\triangleleft}$ *satisfying* $(x \triangleleft y) \,\overline{\triangleleft}\, y = x$ *and* $(x \,\overline{\triangleleft}\, y) \triangleleft y = x$ *iff the right translations of* $(S, \triangleleft)$ *are bijections.*

  - Hence: S-colorings invariant under Reidemeister moves II+III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf with bijective right translations
    ↑ a rack

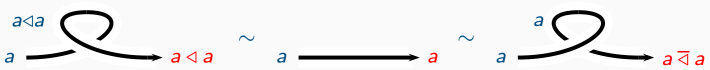
- Idem for Reidemeister move II:



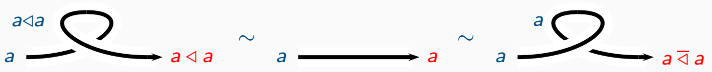
- <u>Lemma</u>: *There exists* $\overline{\triangleleft}$ *satisfying* $(x \triangleleft y) \,\overline{\triangleleft}\, y = x$ *and* $(x \,\overline{\triangleleft}\, y) \triangleleft y = x$ *iff the right translations of* $(S, \triangleleft)$ *are bijections.*
    - Hence: S-colorings invariant under Reidemeister moves II+III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf with bijective right translations
      ↑
      a rack

- <u>Proposition</u>: *Whenever* $(S, \triangleleft)$ *is a rack, diagram coloring provides a well defined action of the braid group* B_n *on* S^n *for every* n.

- Idem for Reidemeister move I:

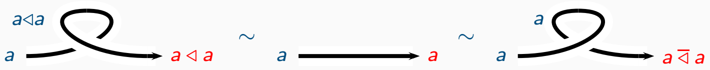


- Idem for Reidemeister move I:



  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack

- Idem for Reidemeister move I:



  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    $\uparrow$
    a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

- Idem for Reidemeister move I:

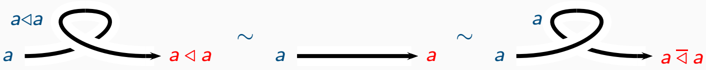
  - Hence: S-colorings invariant under Reidemeister moves I+II+III ⇔ $(S, \triangleleft)$ is an idempotent rack
    ↑ a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Idem for Reidemeister move I:



  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$

    $(S, \triangleleft)$ is an idempotent rack

    ↑

    a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

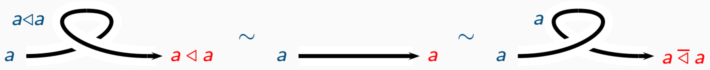
$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:

The trefoil knot:

- Idem for Reidemeister move I:



  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑ a quandle

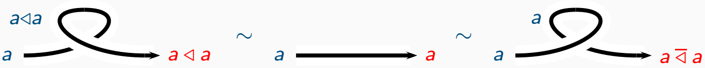
- <u>Theorem</u> (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \dots, a_n \mid a_1 = a'_1, \dots, a_n = a'_n \rangle_{\text{quandle}}$$

where $a'_1, \dots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \dots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- <u>Example</u>:
The trefoil knot:

- Idem for Reidemeister move I:



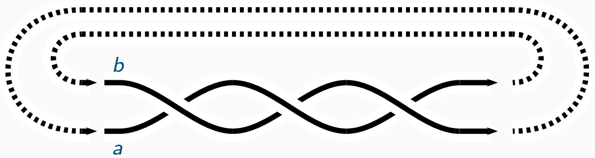
  - Hence: S-colorings invariant under Reidemeister moves I+II+III ⇔ $(S, \triangleleft)$ is an idempotent rack
    ↑ a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

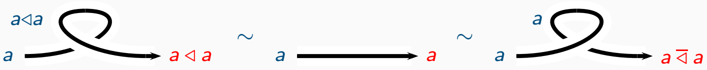
$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
The trefoil knot:



- Idem for Reidemeister move I:



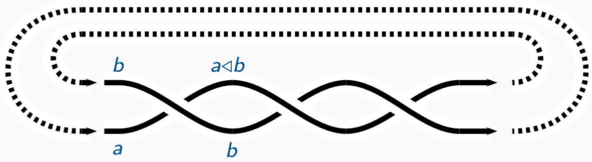
  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑
    a quandle

- <u>Theorem</u> (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- <u>Example</u>:
The trefoil knot:



- Idem for Reidemeister move I:

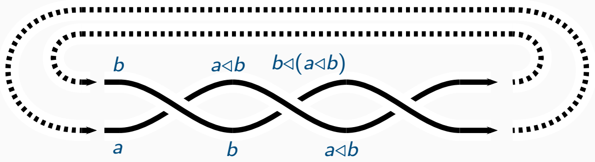
  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑ a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

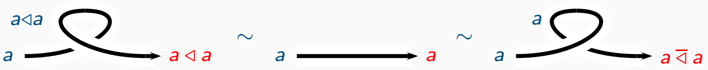
$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{\text{quandle}}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
The trefoil knot:



- Idem for Reidemeister move I:



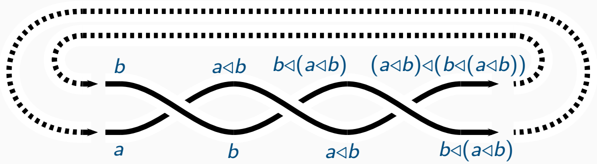
  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑ a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

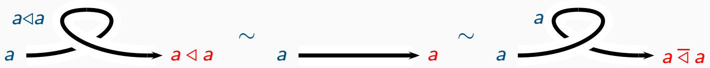
$$\langle a_1, \dots, a_n \mid a_1 = a'_1, \dots, a_n = a'_n \rangle_{\text{quandle}}$$

where $a'_1, \dots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \dots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
The trefoil knot:



- Idem for Reidemeister move I:



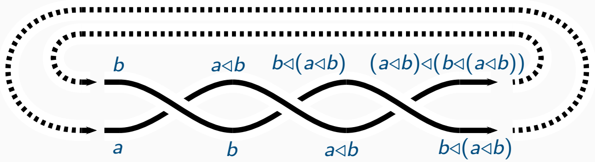
  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑
    a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \dots, a_n \mid a_1 = a'_1, \dots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \dots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \dots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
The trefoil knot:



Leads to $\langle a, b \mid (b \triangleleft (a \triangleleft b) = a, (a \triangleleft b) \triangleleft (b \triangleleft (a \triangleleft b)) = b \rangle_{\text{quandle}}$,

- Idem for Reidemeister move I:

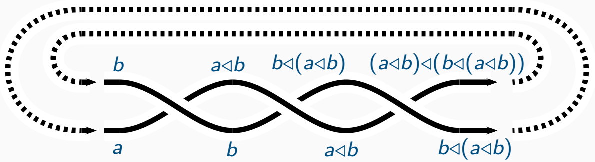
  - Hence: S-colorings invariant under Reidemeister moves I+II+III ⇔
    $(S, \triangleleft)$ is an idempotent rack
    ↑
    a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \ldots, a_n \mid a_1 = a_1', \ldots, a_n = a_n' \rangle_{quandle}$$

where $a_1', \ldots, a_n'$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
  The trefoil knot:



Leads to $\langle a, b \mid (b \triangleleft (a \triangleleft b) = a, (a \triangleleft b) \triangleleft (b \triangleleft (a \triangleleft b)) = b \rangle_{\text{quandle}}$,

i.e., $\langle a, b, c \mid a \triangleleft b = c, b \triangleleft c = a, c \triangleleft a = b \rangle_{\text{quandle}}$.

- Quandles and racks have being used successfully in knot theory

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...
  - Presumably much work to adapt the results. (?)

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...
  - Presumably much work to adapt the results. (?)
- Good news: General shelves are very different from racks.

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...
  - Presumably much work to adapt the results. (?)
- Good news: General shelves are very different from racks.
  - If general shelves can be used, one can expect really new applications.

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...
  - Presumably much work to adapt the results. (?)
- Good news: General shelves are very different from racks.
  - If general shelves can be used, one can expect really new applications.

  - Explore the world of shelves...

- An example (of using non-rack shelves): the partial action of braids
on a right-cancellative shelf

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

$\uparrow$

$a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

and

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

and

the unique c s.t. $c \triangleleft a = b$, if it exists.

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

b, a  $a \triangleleft b$, b and b, a  a, the unique c s.t. $c \triangleleft a = b$, if it exists.

- Proposition: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf
  ↑
  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

b, a  $a \triangleleft b$, b

and

b, a  a, the unique c s.t. $c \triangleleft a = b$, if it exists.

- Proposition: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*
  - *For all n-strand braid words $w_1, \ldots, w_p$, there exists at least one sequence $\vec{a}$ in S^n s.t. $\vec{a} \bullet w_i$ is defined for each i.*

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf
  $\uparrow$
  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

b, a  $a \triangleleft b$, b

and

b, a  a, the unique c s.t. $c \triangleleft a = b$, if it exists.

- Proposition: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*
  - *For all n-strand braid words $w_1, ..., w_p$, there exists at least one sequence $\vec{a}$ in S^n s.t. $\vec{a} \bullet w_i$ is defined for each i.*
  - *If w, w' are equivalent n-strand braid words and $\vec{a} \bullet w$ and $\vec{a} \bullet w'$ are defined, then $\vec{a} \bullet w = \vec{a} \bullet w'$ holds.*

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

b, a  $a \triangleleft b$, b

and

b, a  a, the unique c s.t. $c \triangleleft a = b$, if it exists.

- Proposition: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*
  - *For all n-strand braid words $w_1, \ldots, w_p$, there exists at least one sequence $\vec{a}$ in S^n s.t. $\vec{a} \bullet w_i$ is defined for each i.*
  - *If w, w' are equivalent n-strand braid words and $\vec{a} \bullet w$ and $\vec{a} \bullet w'$ are defined, then $\vec{a} \bullet w = \vec{a} \bullet w'$ holds.*

  - Proof: Not trivial, uses the Garside structure of braids. □

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

and

the unique c s.t. $c \triangleleft a = b$, if it exists.

- <u>Proposition</u>: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*
  - *For all n-strand braid words $w_1, \ldots, w_p$, there exists at least one sequence $\vec{a}$ in S^n s.t. $\vec{a} \bullet w_i$ is defined for each i.*
  - *If w, w' are equivalent n-strand braid words and $\vec{a} \bullet w$ and $\vec{a} \bullet w'$ are defined, then $\vec{a} \bullet w = \vec{a} \bullet w'$ holds.*

  - Proof: Not trivial, uses the Garside structure of braids. □

$\rightsquigarrow$ a usable partial action...

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.

$$a < b \text{ implies } a \triangleleft c < b \triangleleft c,$$

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
$a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack.

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $b \triangleleft (a \triangleleft a)$

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a)$

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a$

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $b \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft a = b \triangleleft a,$

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$.

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative:

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,
    whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$,

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
  $$b \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
  hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$, whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $b \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \overline{\triangleleft} a) \triangleleft a) \triangleleft a = b \triangleleft a$,
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$, whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:

- Definition: A shelf is **orderable** if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,
    whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

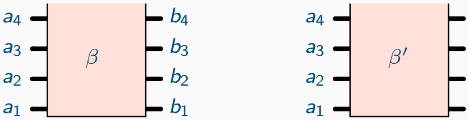
- Coloring braids using an orderable shelf directly provides a linear ordering of braids:

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,
    whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$, whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:

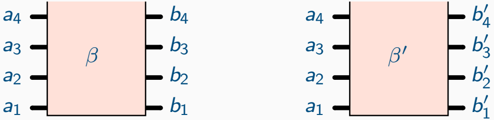


- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \mathbin{\overline{\triangleleft}} a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,
    whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t. $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$, whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:



  - Then define $\beta < \beta'$ iff $\vec{a} \bullet \beta <^{\mathrm{Lex}} \vec{a} \bullet \beta'$.
    $\uparrow$
    $(b_1 < b'_1)$ or $(b_1 = b'_1$ and $b_2 < b'_2)$ or etc.

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

- “Trivial” shelves:

- "Trivial" shelves: S a set, f a map $S \to S$,

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \lhd y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \lhd q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \lhd q := p + 1$.

- Lattice shelves:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p+1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p+1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice,

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x \ (= x)$.
  - A non-idempotent related example: B a Boolean algebra,

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$. (i.e., "$x \Leftarrow y$")

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.

    (i.e., "$x \Leftarrow y$")

- Alexander shelves:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p+1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p+1$.
- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")
- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1-t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p+1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p+1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x \ (= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$. (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module, and $x \triangleleft y := (1-t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x \ (= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$. (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module, and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x \; (= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$. (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module, and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.

    ↑ when viewed as $(Q, \triangleleft, \overline{\triangleleft})$:

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.
    ↑
    when viewed as $(Q, \triangleleft, \overline{\triangleleft})$: $(F_X, \triangleleft)$ is not a free idempotent shelf,
    it satisfies other laws: $x \triangleleft (y \triangleleft (y \triangleleft x)) = (x \triangleleft (x \triangleleft y)) \triangleleft (y \triangleleft x)$, ...
    (Drápal-Kepka-Musílek, Larue)

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \lhd y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \lhd q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \lhd q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \lhd y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \lhd x = x \lhd x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \lhd y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \lhd y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \lhd y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \lhd y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.
    $\uparrow$
    when viewed as $(Q, \lhd, \overline{\lhd})$: $(F_X, \lhd)$ is not a free idempotent shelf,
    it satisfies other laws: $x \lhd (y \lhd (y \lhd x)) = (x \lhd (x \lhd y)) \lhd (y \lhd x)$, ...
    (Drápal-Kepka-Musílek, Larue)
  - Variants: $x \lhd y := y^{-n}xy^n$,

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p+1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p+1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module,
  and $x \triangleleft y := (1-t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.
    ↑
    when viewed as $(Q, \triangleleft, \overline{\triangleleft})$: $(F_X, \triangleleft)$ is not a free idempotent shelf,
    it satisfies other laws: $x \triangleleft (y \triangleleft (y \triangleleft x)) = (x \triangleleft (x \triangleleft y)) \triangleleft (y \triangleleft x)$, ...
    (Drápal-Kepka-Musílek, Larue)
  - Variants: $x \triangleleft y := y^{-n}xy^n$, $x \triangleleft y := f(y^{-1}x)y$ with $f \in \mathrm{Aut}(G)$, ...

- Core (or sandwich) quandles:

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks:

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf:

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

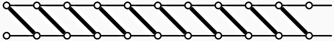
- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with sh : $n \mapsto n + 1$:

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

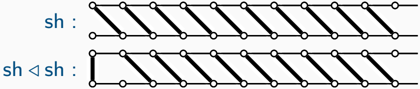
- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n + 1$:

sh : 

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

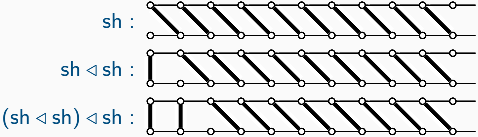
- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n+1$:



- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

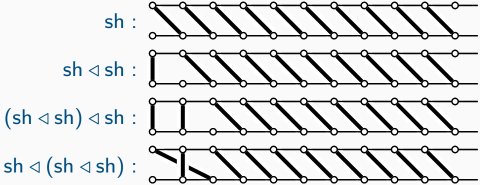
- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n + 1$:



- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G, and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself, and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n+1$:



- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

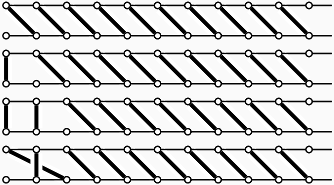
- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n+1$:

sh :

$\mathrm{sh} \triangleleft \mathrm{sh}$:

$(\mathrm{sh} \triangleleft \mathrm{sh}) \triangleleft \mathrm{sh}$:

$\mathrm{sh} \triangleleft (\mathrm{sh} \triangleleft \mathrm{sh})$:



[P.D. *Algebraic properties of the shift mapping*, Proc. Amer. Math. Soc. 106 (1989) 617-623]

- Braid shelf: B_∞ braid group,

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$,

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.

- Braid shelf: B_∞ braid group, $\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathsf{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha) \cdot \beta$.
  - Part 3 below

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

[P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with ⩾2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant$2 generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory):

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant$2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant$2 generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself,

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

- Laver tables:

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with ⩾2 generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of ⩾2 generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

- Laver tables: a family of finite shelves with 2^n elements

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

- Laver tables: a family of finite shelves with 2^n elements
  - A. Drápal's minicourse

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

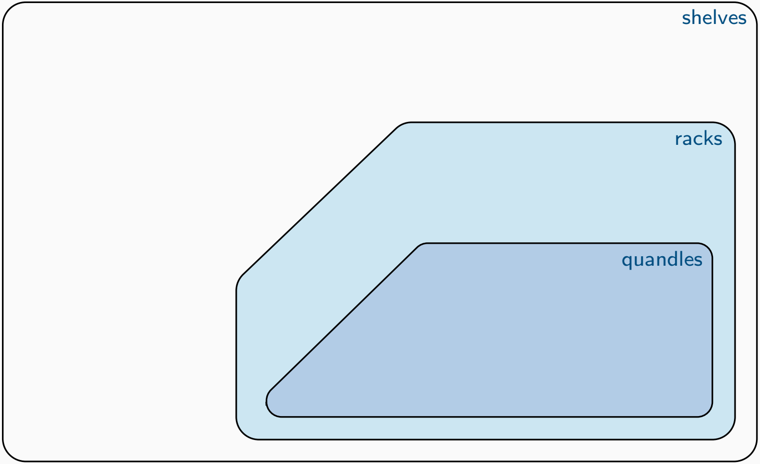
Transfinite braids and left distributive operations;

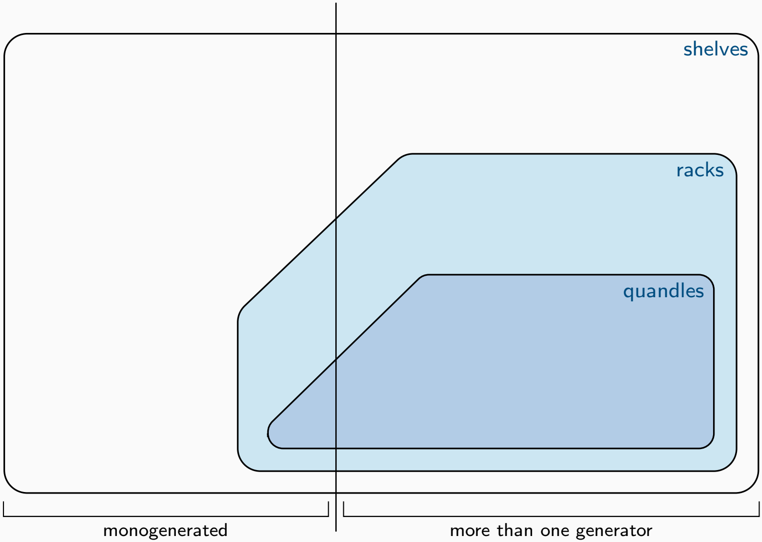
- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

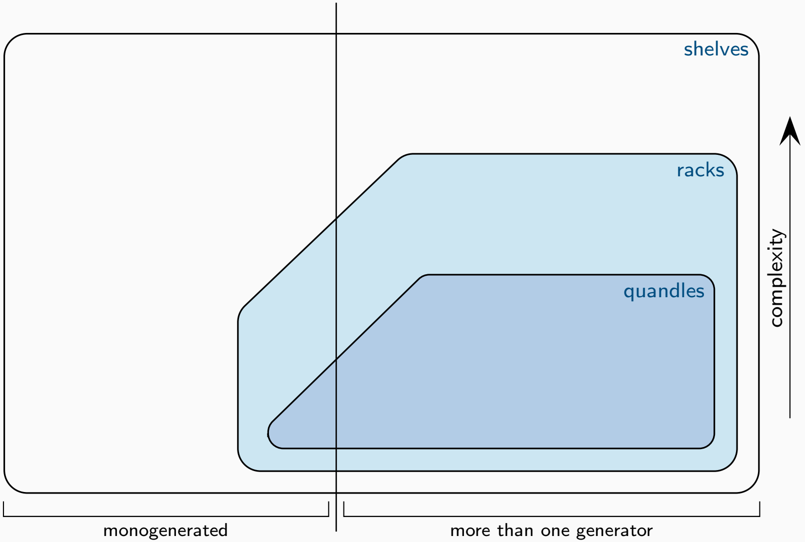
    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

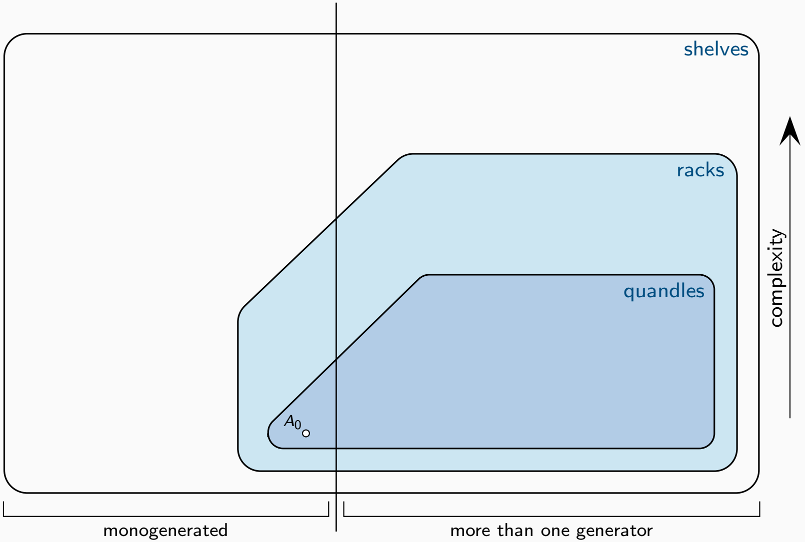
- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

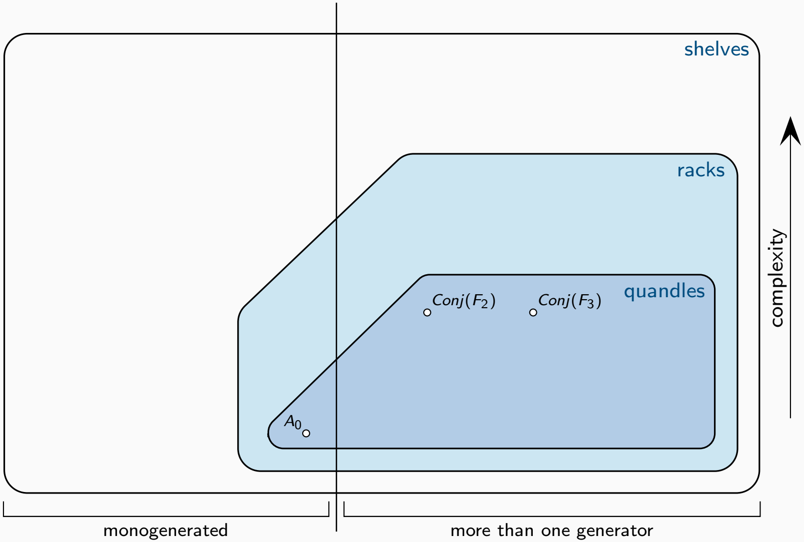
- Laver tables: a family of finite shelves with 2^n elements
  - A. Drápal's minicourse
  - Part 6 below

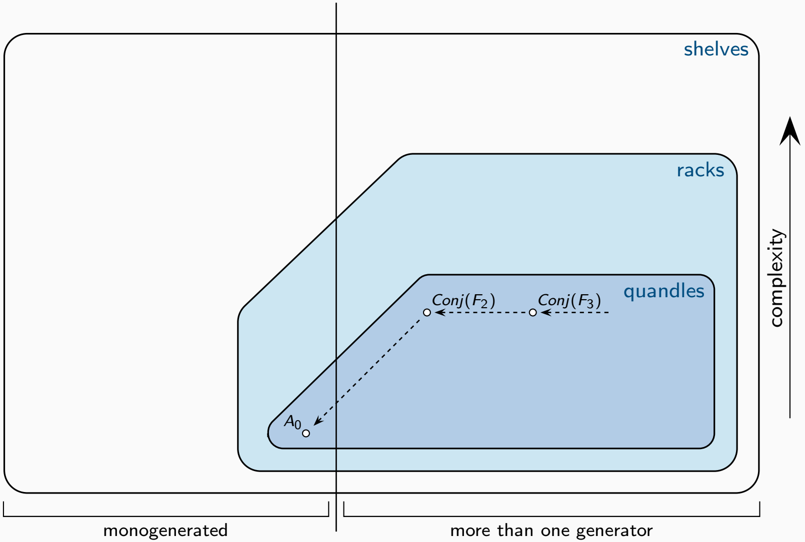

shelves
racks
quandles

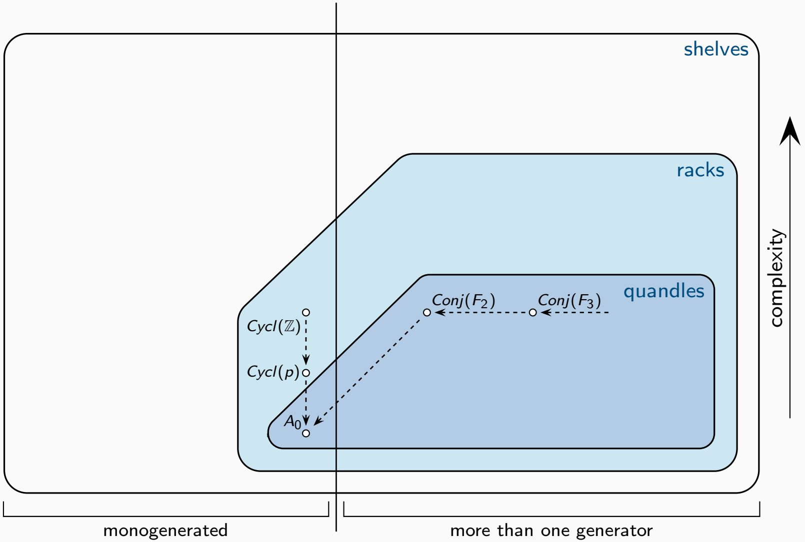

shelves
racks
quandles
monogenerated
more than one generator

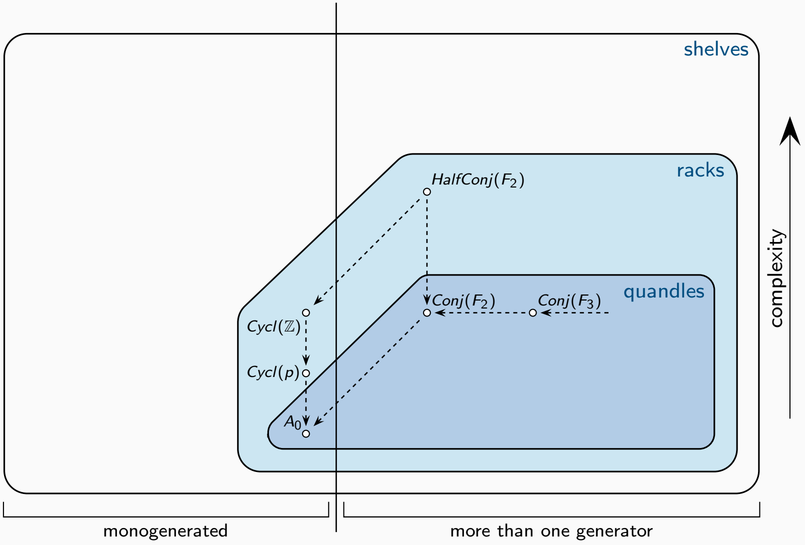

shelves
racks
quandles
complexity
monogenerated
more than one generator

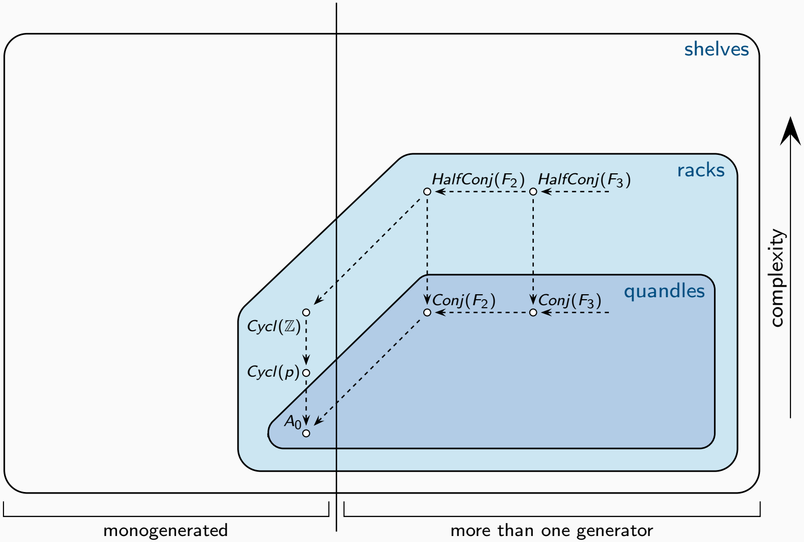

shelves
racks
quandles
A_0
complexity
monogenerated
more than one generator

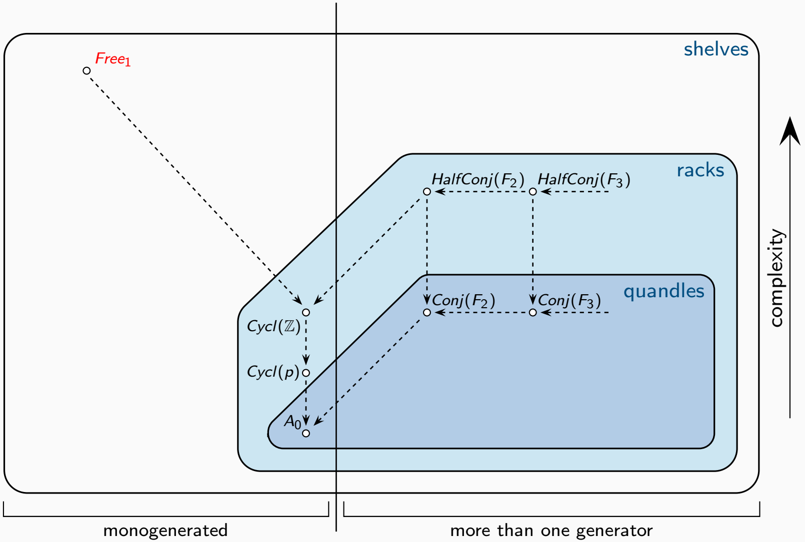

shelves
racks
quandles
Conj(F2)
Conj(F3)
A0
complexity
monogenerated
more than one generator

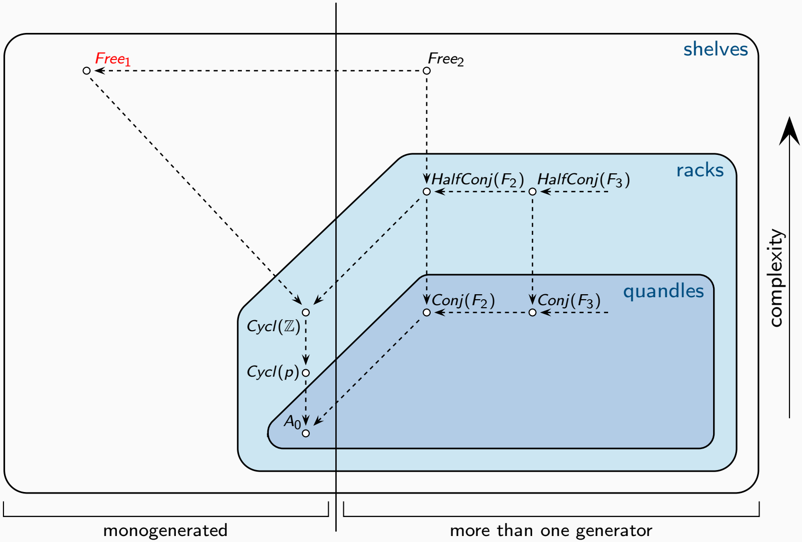

shelves
racks
quandles
$Conj(F_2)$
$Conj(F_3)$
A_0
complexity
monogenerated
more than one generator

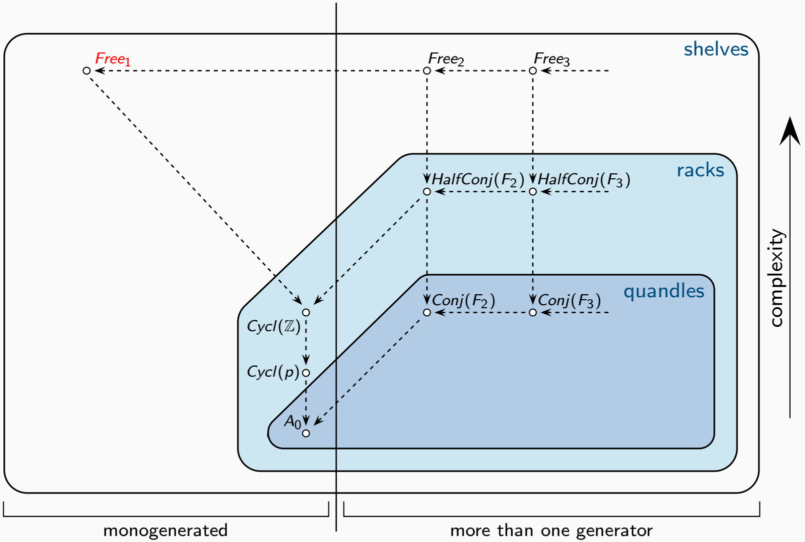

shelves
racks
quandles
complexity
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
$Conj(F_2)$
$Conj(F_3)$
monogenerated
more than one generator

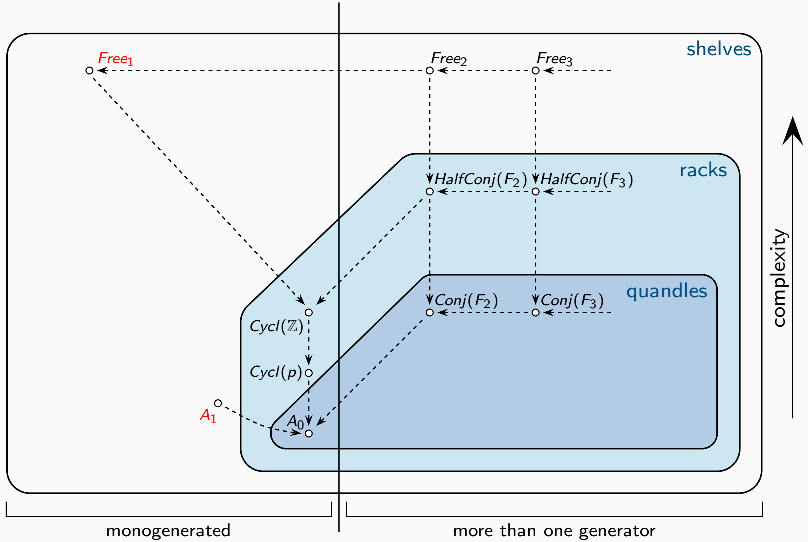

shelves
racks
quandles
$HalfConj(F_2)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
$Conj(F_2)$
$Conj(F_3)$
complexity
monogenerated
more than one generator

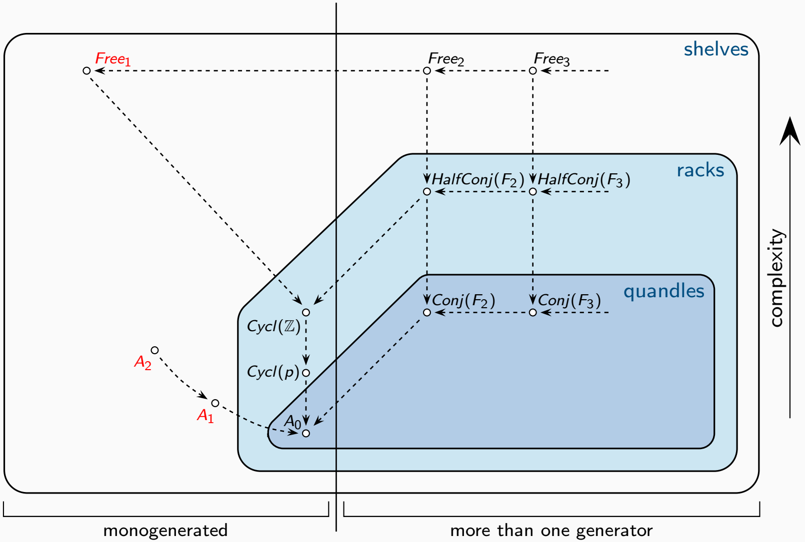

shelves
racks
quandles
HalfConj(F_2)
HalfConj(F_3)
Conj(F_2)
Conj(F_3)
Cycl($\mathbb{Z}$)
Cycl(p)
A_0
complexity
monogenerated
more than one generator

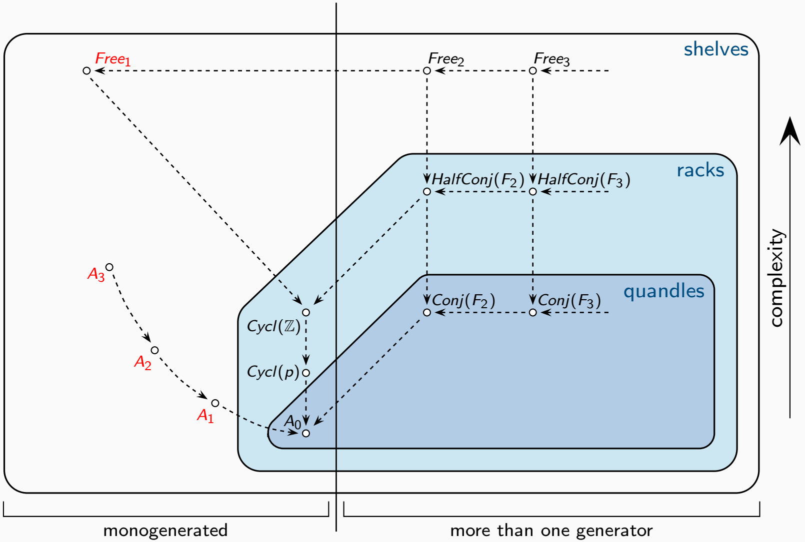

shelves
Free_1
racks
HalfConj(F_2)
HalfConj(F_3)
quandles
Conj(F_2)
Conj(F_3)
Cycl(ℤ)
Cycl(p)
A_0
complexity
monogenerated
more than one generator

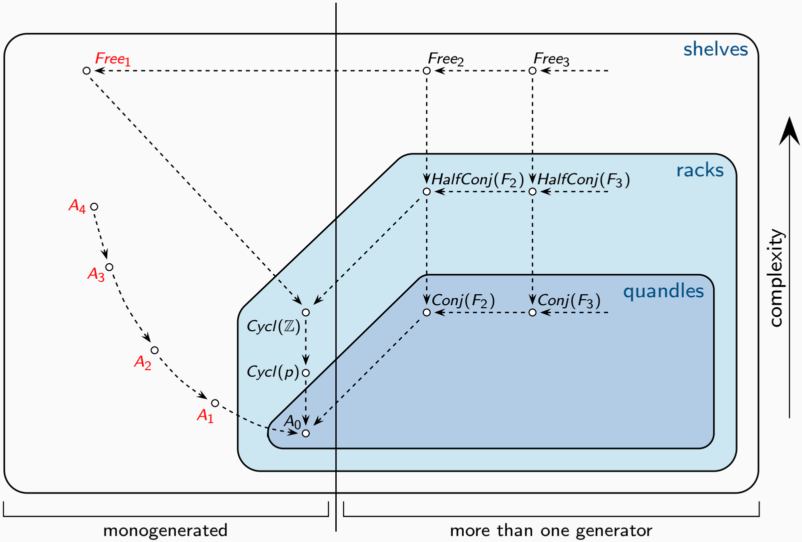

shelves
Free$_1$
Free$_2$
racks
HalfConj(F_2)
HalfConj(F_3)
quandles
Conj(F_2)
Conj(F_3)
Cycl$(\mathbb{Z})$
Cycl(p)
A_0
complexity
monogenerated
more than one generator

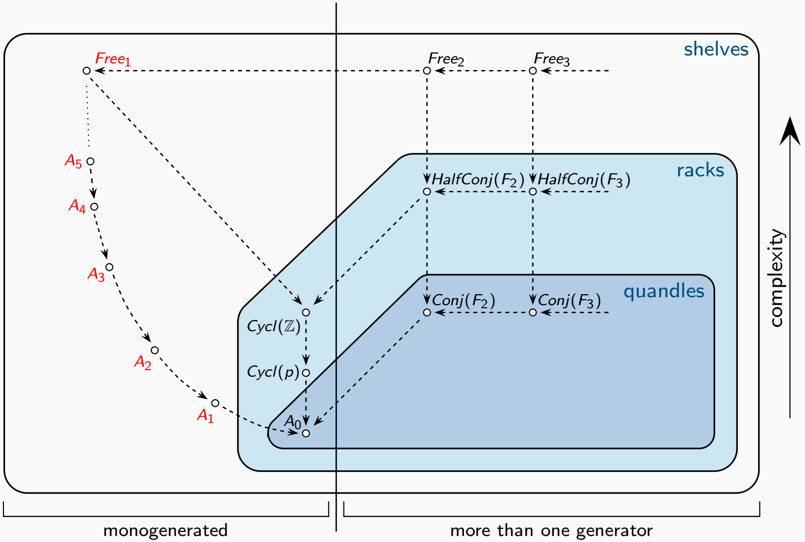

shelves
racks
quandles
Free1
Free2
Free3
HalfConj(F2)
HalfConj(F3)
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
A0
complexity
monogenerated
more than one generator

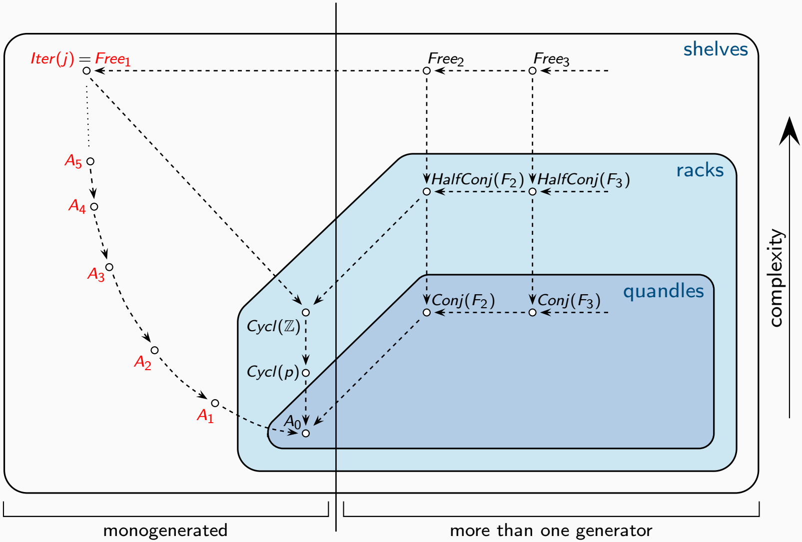

shelves
racks
quandles
$Free_1$
$Free_2$
$Free_3$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_1
A_0
complexity
monogenerated
more than one generator

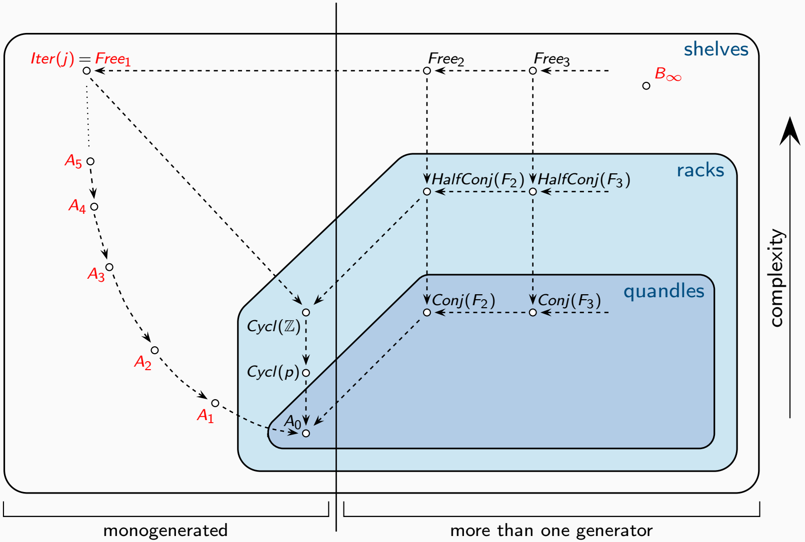

shelves
$Free_1$
$Free_2$
$Free_3$
racks
$HalfConj(F_2)$
$HalfConj(F_3)$
quandles
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_2
A_1
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
complexity
Free1
Free2
Free3
HalfConj(F2)
HalfConj(F3)
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
A0
A1
A2
A3
monogenerated
more than one generator

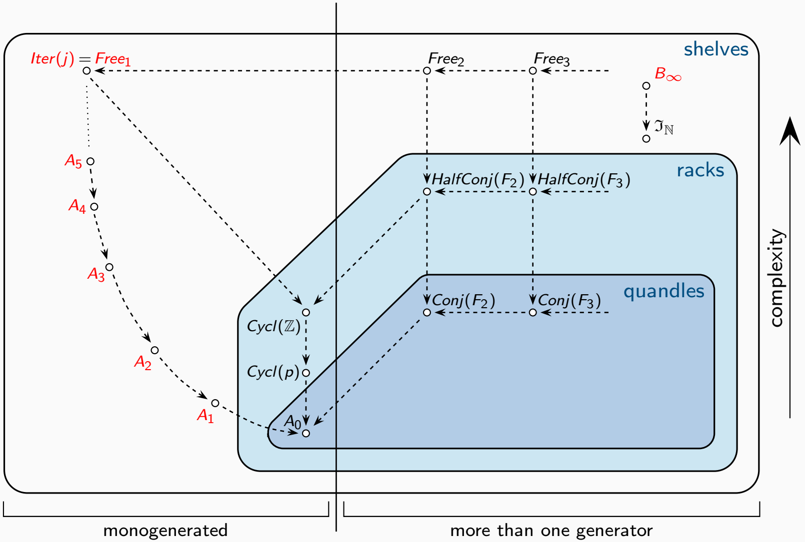

shelves
racks
quandles
$Free_1$
$Free_2$
$Free_3$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_4
A_3
A_2
A_1
A_0
complexity
monogenerated
more than one generator

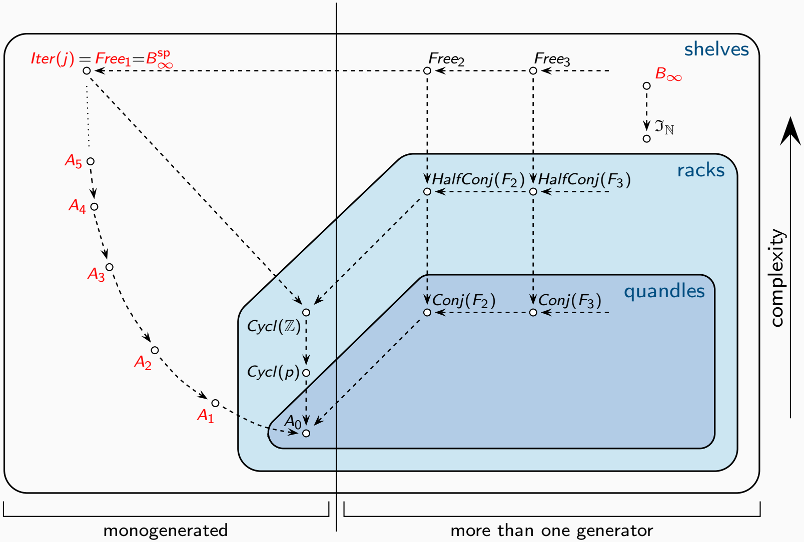

shelves
Free1
Free2
Free3
racks
HalfConj(F2)
HalfConj(F3)
quandles
Conj(F2)
Conj(F3)
A5
A4
A3
A2
A1
A0
Cycl(ℤ)
Cycl(p)
complexity
monogenerated
more than one generator


shelves
racks
quandles
Iter(j) = Free1
Free2
Free3
A5
A4
A3
A2
A1
A0
HalfConj(F2)
HalfConj(F3)
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
complexity
monogenerated
more than one generator

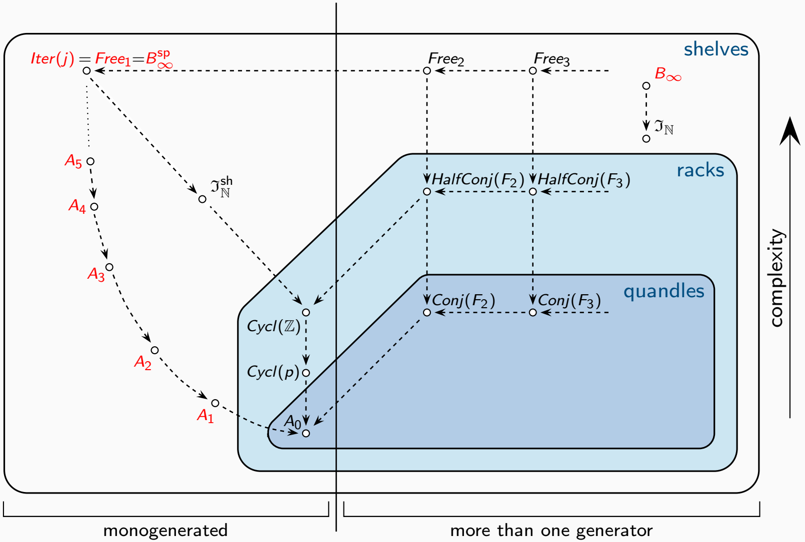


shelves
$Iter(j) = Free_1$
$Free_2$
$Free_3$
B_∞
A_5
A_4
A_3
A_2
A_1
A_0
racks
$HalfConj(F_2)$
$HalfConj(F_3)$
quandles
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator


shelves
Iter(j) = Free1
Free2
Free3
B∞
ℑℕ
A5
A4
A3
A2
A1
A0
racks
HalfConj(F2)
HalfConj(F3)
quandles
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
complexity
monogenerated
more than one generator



Iter(j) = Free1=B∞sp
Free2
Free3
shelves
B∞
𝔍ℕ
A5
A4
A3
A2
A1
A0
racks
HalfConj(F2)
HalfConj(F3)
quandles
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Iter(j) = Free_1 = B_\infty^{sp}$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
$\mathfrak{I}_\mathbb{N}^{sh}$
A_5
A_4
A_3
A_2
A_1
A_0
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \qquad\qquad\qquad\qquad \right\rangle.$$

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| \geqslant 2 \right\rangle.$$

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

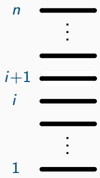
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

- <u>Definition</u> (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

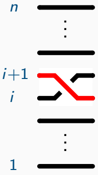


- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$

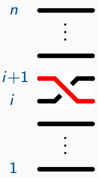


- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$



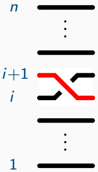
$\simeq$ mapping class group of D_n (disk with n punctures):

- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

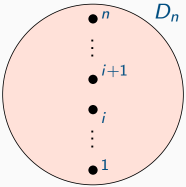
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$



$\simeq$ mapping class group of D_n (disk with n punctures):

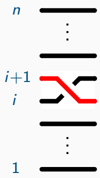


- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

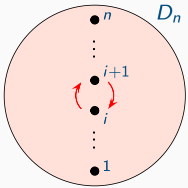
$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$



$\simeq$ mapping class group of D_n (disk with n punctures):

$\sigma_i \quad \leftrightarrow$



- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit B_∞

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \right\rangle$.

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2 \right\rangle$.

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{matrix} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathrm{sh}(\beta) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}.$$

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathrm{sh}(\beta) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}.$$

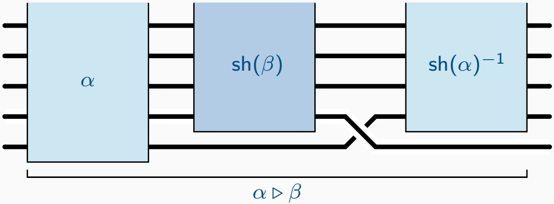
Then $(B_\infty, \triangleright)$ is a left shelf.

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathrm{sh}(\beta) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.

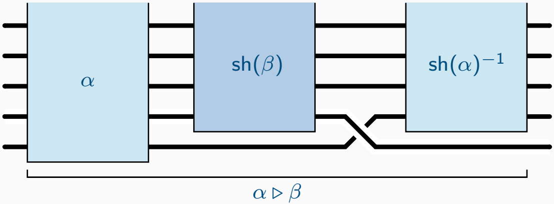


- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathsf{sh}(\beta) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



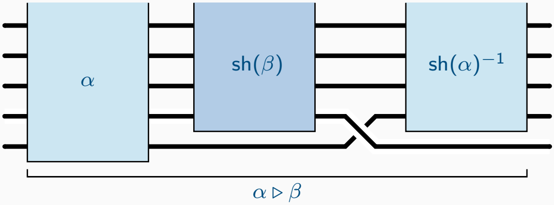
- Examples: $1 \triangleright 1 = \sigma_1$,

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathrm{sh}(\beta) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



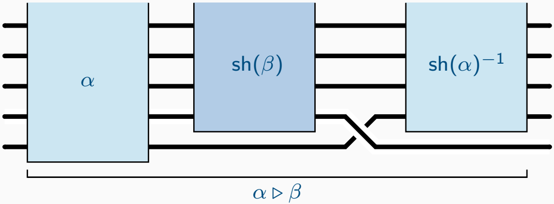
- Examples: $1 \triangleright 1 = \sigma_1$, $1 \triangleright \sigma_1 = \sigma_2\sigma_1$,

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \dots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathsf{sh}(\beta) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



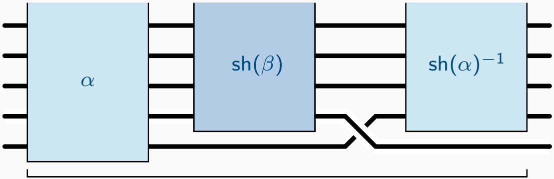
- Examples: $1 \triangleright 1 = \sigma_1$, $\quad 1 \triangleright \sigma_1 = \sigma_2\sigma_1$, $\quad \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$,

- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{matrix} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathsf{sh}(\beta) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



$\alpha \triangleright \beta$

- Examples: $1 \triangleright 1 = \sigma_1$, $\quad 1 \triangleright \sigma_1 = \sigma_2\sigma_1$, $\quad \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$, $\quad \sigma_1 \triangleright \sigma_1 = \sigma_2\sigma_1$, etc.

- Proof:

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) =$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
  $= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \mathsf{sh}(\alpha)^{-1}.$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \mathsf{sh}(\alpha)^{-1}.$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$

$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$$

$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}.$$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$

$$= (\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\gamma)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \; \sigma_1 \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1})^{-1}$$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
  $= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
  $= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \mathsf{sh}(\alpha)^{-1}.$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}) \cdot \mathsf{sh}(\alpha\, \mathsf{sh}(\gamma)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1})^{-1}$
$= \alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}\, \mathsf{sh}(\alpha)\, \mathsf{sh}^2(\gamma)\, \sigma_2\, \mathsf{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathsf{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathsf{sh}^2(\beta)^{-1}\, \mathsf{sh}(\alpha)^{-1}$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathsf{sh}(\beta \cdot \mathsf{sh}(\gamma) \cdot \sigma_1 \cdot \mathsf{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathsf{sh}(\beta) \cdot \mathsf{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathsf{sh}^2(\beta)^{-1} \cdot \mathsf{sh}(\alpha)^{-1}.$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}) \cdot \mathsf{sh}(\alpha\, \mathsf{sh}(\gamma)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1})^{-1}$
$= \alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}(\alpha)^{-1}\, \mathsf{sh}(\alpha)\, \mathsf{sh}^2(\gamma)\, \sigma_2\, \mathsf{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathsf{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathsf{sh}^2(\beta)^{-1}\, \mathsf{sh}(\alpha)^{-1}$
$= \alpha\, \mathsf{sh}(\beta)\, \sigma_1\, \mathsf{sh}^2(\gamma)\, \sigma_2\sigma_1\sigma_2^{-1}\, \mathsf{sh}^2(\beta)^{-1}\, \mathsf{sh}(\alpha)^{-1}$

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}.$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\gamma)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1})^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}\, \mathrm{sh}(\alpha)\, \mathrm{sh}^2(\gamma)\, \sigma_2\, \mathrm{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathrm{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}^2(\gamma)\, \sigma_2\sigma_1\sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$ □

- Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}.$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\,\mathrm{sh}(\gamma)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1})^{-1}$
$= \alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}\,\mathrm{sh}(\alpha)\,\mathrm{sh}^2(\gamma)\,\sigma_2\,\mathrm{sh}^2(\alpha)^{-1}\,\sigma_1\,\mathrm{sh}^2(\alpha)\,\sigma_2^{-1}\,\mathrm{sh}^2(\beta)^{-1}\,\mathrm{sh}(\alpha)^{-1}$
$= \alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}^2(\gamma)\,\sigma_2\sigma_1\sigma_2^{-1}\,\mathrm{sh}^2(\beta)^{-1}\,\mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$ □

- Remark: Shelf (=right shelf) with

$$\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta,$$

▶ Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$

$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$$

$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}.$$

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$

$$= (\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\gamma)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot\ \sigma_1 \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1})^{-1}$$

$$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}\, \mathrm{sh}(\alpha)\, \mathrm{sh}^2(\gamma)\, \sigma_2\, \mathrm{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathrm{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$$

$$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}^2(\gamma)\, \sigma_2\sigma_1\sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$$

$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1} \qquad \square$$

- Remark: Shelf (=right shelf) with

$$\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta,$$

but less convenient here.

▶ Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$.

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\gamma)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot\ \sigma_1 \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1})^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}\, \mathrm{sh}(\alpha)\, \mathrm{sh}^2(\gamma)\, \sigma_2\, \mathrm{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathrm{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}^2(\gamma)\, \sigma_2\sigma_1\sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$ □

- Remark: Shelf (=right shelf) with

$$\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta,$$

but less convenient here.

- Remark: Works similarly with

$$x \triangleright y := x \cdot \phi(y) \cdot e \cdot \phi(x)^{-1}$$

whenever G is a group G, e belongs to G, and ϕ is an endomorphism ϕ satisfying

$$e\,\phi(e)\,e = \phi(e)\,e\,\phi(e) \quad \text{and} \quad \forall x\,(e\,\phi^2(x) = \phi^2(x)\,e).$$

• <u>Proposition</u> (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

• <u>Proposition</u> (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$: $\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$,

• <u>Proposition</u> (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

▶ Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• <u>Theorem</u> (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

▶ Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

▶ Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}, \quad \rho(\sigma_i)(x_{i+1}) := x_i,$

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:

$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}, \quad \rho(\sigma_i)(x_{i+1}) := x_i, \quad \rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Want to prove: $\rho(\alpha) \neq \rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n)$.

• Proposition (D., 1989, Laver, 1989) *If* $(S, \triangleright)$ *is a monogenerated left shelf, a sufficient condition for* $(S, \triangleright)$ *to be free is that the relation* $\sqsubset$ *on* S *has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

▶ Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in* B_∞ *generates in* $(B_\infty, \triangleright)$ *a free left shelf.*

▶ Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

▶ Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Want to prove: $\rho(\alpha) \neq \rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n)$.
By definition: $\rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n) = \rho(\alpha) \circ \rho(\gamma)$,
with γ a braid of the form $\mathrm{sh}(\gamma_0)\, \sigma_1\, \mathrm{sh}(\gamma_1)\, \sigma_1\, \mathrm{sh}(\gamma_2) \cdots \sigma_1\, \mathrm{sh}(\gamma_n)$, with no σ_1^{-1}.

• Proposition (D., 1989, Laver, 1989) *If $(S, \triangleright)$ is a monogenerated left shelf, a sufficient condition for $(S, \triangleright)$ to be free is that the relation $\sqsubset$ on S has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

▶ Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in B_∞ generates in $(B_\infty, \triangleright)$ a free left shelf.*

▶ Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

▶ Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Want to prove: $\rho(\alpha) \neq \rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n)$.
By definition: $\rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n) = \rho(\alpha) \circ \rho(\gamma)$,
with γ a braid of the form $\mathrm{sh}(\gamma_0)\, \sigma_1\, \mathrm{sh}(\gamma_1)\, \sigma_1\, \mathrm{sh}(\gamma_2) \cdots \sigma_1\, \mathrm{sh}(\gamma_n)$, with no σ_1^{-1}.
Call such a braid σ_1-positive.

• Proposition (D., 1989, Laver, 1989) *If $(S, \triangleright)$ is a monogenerated left shelf, a sufficient condition for $(S, \triangleright)$ to be free is that the relation $\sqsubset$ on S has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\, (x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in B_∞ generates in $(B_\infty, \triangleright)$ a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Want to prove: $\rho(\alpha) \neq \rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n)$.
By definition: $\rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n) = \rho(\alpha) \circ \rho(\gamma)$,
with γ a braid of the form $\mathrm{sh}(\gamma_0)\, \sigma_1\, \mathrm{sh}(\gamma_1)\, \sigma_1\, \mathrm{sh}(\gamma_2) \cdots \sigma_1\, \mathrm{sh}(\gamma_n)$, with no σ_1^{-1}.
Call such a braid σ_1-positive. It suffices to prove: "β is σ_1-positive $\Rightarrow \rho(\beta) \neq \mathrm{id}_{F_\infty}$".

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  - Proof:

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

  - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \ldots\}$ (no ss^{-1} or $s^{-1}s$).

- <u>Lemma</u> (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

  - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$.

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}*.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1))$

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \dots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1)$

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}*.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1}$

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.
Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$?

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.
  Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \dots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
  Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

   - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \ldots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
   $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
   If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
   If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
   So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.
   Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
   Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}.

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \dots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$. Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \ldots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
  Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

  - For $w' = u x_1 v$, we find
    $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \dots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.

Hence $\phi(v) = 1$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
  Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

  - For $w' = ux_1v$, we find
    $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.

  Hence $\phi(v) = 1$, then $v = 1$,

- <u>Lemma</u> (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$, we find
  $\phi(w) = \mathrm{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\mathrm{red}(x_2 x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.

Hence $\phi(v) = 1$, then $v = 1$, and $w' = u x_1$,

• <u>Lemma</u> (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.

If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.

So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.

Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

- For $w' = ux_1^{-1}v$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
  Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

  - For $w' = ux_1v$, we find
    $\phi(w) = \operatorname{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\operatorname{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

  - For $w' = ux_1^{-1}v$, we find
    $\phi(w) = \operatorname{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w') x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2 x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = 1$, then $v = 1$, and $w' = u x_1$, contradicting "$w' x_1^{-1}$ reduced".

- For $w' = u x_1^{-1} v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, \ldots\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

- For $w' = ux_1^{-1}v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_1x_2^2x_1^{-1}$,

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

- For $w' = ux_1^{-1}v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
Hence $\phi(v) = x_1x_2^2x_1^{-1}$, then $v = x_1^2$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2 x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.

Hence $\phi(v) = 1$, then $v = 1$, and $w' = u x_1$, contradicting "$w' x_1^{-1}$ reduced".

- For $w' = u x_1^{-1} v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.

Hence $\phi(v) = x_1 x_2^2 x_1^{-1}$, then $v = x_1^2$, and $w' = u x_1^{-1} x_1^2$, contradicting "w' reduced".

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \operatorname{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\operatorname{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".
- For $w' = ux_1^{-1}v$, we find
  $\phi(w) = \operatorname{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\operatorname{red}(x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_1x_2^2x_1^{-1}$, then $v = x_1^2$, and $w' = ux_1^{-1}x_1^2$, contradicting "w' reduced".
- For $w' = ux_2v$,

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  - Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2 x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = 1$, then $v = 1$, and $w' = u x_1$, contradicting "$w' x_1^{-1}$ reduced".

- For $w' = u x_1^{-1} v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = x_1 x_2^2 x_1^{-1}$, then $v = x_1^2$, and $w' = u x_1^{-1} x_1^2$, contradicting "w' reduced".

- For $w' = u x_2 v$, we find
  $\phi(w) = \operatorname{red}(\phi(u) x_1 \phi(v) x_1 x_2^{-1} x_1^{-1})$ with $\operatorname{red}(\phi(v) x_1 x_2^{-1}) = 1$.

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

  ▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let
  $$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$
  If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
  If $\beta = \sigma_1\,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
  So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

  Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
  Then $\phi(w) = \mathrm{red}(\phi(w')\,x_1x_2^{-1}x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

  - For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

  - For $w' = ux_1^{-1}v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_1x_2^2x_1^{-1}$, then $v = x_1^2$, and $w' = ux_1^{-1}x_1^2$, contradicting "w' reduced".

  - For $w' = ux_2v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1\phi(v)x_1x_2^{-1}x_1^{-1})$ with $\mathrm{red}(\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_2^{-1}x_1$, then $v = x_2^{-1}x_1$, and $w' = ux_2x_2^{-1}x_1$, contradicting "w' reduced".□

- Lemma (Larue, 1992) *If* β *is* σ_1*-positive, then* $\rho(\beta)(x_1)$ *finishes with* x_1^{-1}.

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \operatorname{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\operatorname{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1 x_2 x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w' x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \operatorname{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = u x_1 v$, we find
$\phi(w) = \operatorname{red}(\phi(u) x_1 x_2 x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2 x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = 1$, then $v = 1$, and $w' = u x_1$, contradicting "$w' x_1^{-1}$ reduced".

- For $w' = u x_1^{-1} v$, we find
$\phi(w) = \operatorname{red}(\phi(u) x_1 x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1} x_1^{-1})$, with $\operatorname{red}(x_2^{-1} x_1^{-1} \phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = x_1 x_2^2 x_1^{-1}$, then $v = x_1^2$, and $w' = u x_1^{-1} x_1^2$, contradicting "w' reduced".

- For $w' = u x_2 v$, we find
$\phi(w) = \operatorname{red}(\phi(u) x_1 \phi(v) x_1 x_2^{-1} x_1^{-1})$ with $\operatorname{red}(\phi(v) x_1 x_2^{-1}) = 1$.
Hence $\phi(v) = x_2^{-1} x_1$, then $v = x_2^{-1} x_1$, and $w' = u x_2 x_2^{-1} x_1$, contradicting "w' reduced".□

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special;

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special;

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special;

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids.*

- <u>Definition</u>: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1\sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$ is special, etc.

- <u>Proposition</u>: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent*

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
    - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, ...) \bullet \beta = (\beta, 1, 1, ...).$$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special;

$(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special.

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1.

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1. Then
    $(1, 1, \ldots) \bullet (\alpha \triangleright \beta) =$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1. Then $(1, 1, \ldots) \bullet (\alpha \triangleright \beta) = (((1, 1, \ldots) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1}$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1\sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, ...) \bullet \beta = (\beta, 1, 1, ...).$$

  - Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, ...) \bullet (\alpha \triangleright \beta) &= (((1, 1, ...) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, ...) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1}\end{aligned}$$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, \ldots) \bullet (\alpha \triangleright \beta) &= (((1, 1, \ldots) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, \ldots) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha, \beta, 1, \ldots) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1}\end{aligned}$$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, \ldots) \bullet (\alpha \triangleright \beta) &= (((1, 1, \ldots) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, \ldots) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha, \beta, 1, \ldots) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, \alpha, 1, \ldots) \bullet \mathrm{sh}(\alpha)^{-1}\end{aligned}$$

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.

  ▶ Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  ▶ Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, \ldots) \bullet (\alpha \triangleright \beta) &= (((1, 1, \ldots) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, \ldots) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha, \beta, 1, \ldots) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, \alpha, 1, \ldots) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, 1, 1, \ldots)\end{aligned}$$

□

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1 \sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2 \sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, \ldots) \bullet \beta = (\beta, 1, 1, \ldots).$$

  - Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, \ldots) \bullet (\alpha \triangleright \beta) &= (((1, 1, \ldots) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, \ldots) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha, \beta, 1, \ldots) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, \alpha, 1, \ldots) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, 1, 1, \ldots)\end{aligned}$$

□

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$.

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathsf{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \mathsf{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathsf{sh}} \vec{\alpha} \cdot \beta = \prod^{\mathsf{sh}} \vec{\gamma}.$$

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathsf{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \mathsf{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathsf{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathsf{sh}}\vec{\gamma}.$$

  - Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$.

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathsf{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \mathsf{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathsf{sh}} \vec{\alpha} \cdot \beta = \prod^{\mathsf{sh}} \vec{\gamma}.$$

  - Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$,

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \text{implies} \quad \prod^{\mathrm{sh}} \vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}} \vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$,

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \text{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

• Lemma: *For* $\vec\alpha = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathsf{sh}}\vec\alpha$ *for* $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \mathsf{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec\alpha \bullet \beta = \vec\gamma \quad \textit{implies} \quad \prod^{\mathsf{sh}}\vec\alpha \cdot \beta = \prod^{\mathsf{sh}}\vec\gamma.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec\alpha$ contributes $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \cdots$, whereas $\vec\gamma$ contributes
$\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha_1)^{-1} \cdot \mathsf{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathsf{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathsf{sh}(\beta_2) \cdot \mathsf{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$.

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{sh}\vec{\alpha}$ *for* $\alpha_1 \cdot sh(\alpha_2) \cdot sh^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{sh}\vec{\alpha} \cdot \beta = \prod^{sh}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot sh(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot sh(\alpha_1)^{-1} \cdot sh(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $sh^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot sh(\beta_2) \cdot sh^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{sh}\vec{\alpha}$ *for* $\alpha_1 \cdot sh(\alpha_2) \cdot sh^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{sh}\vec{\alpha} \cdot \beta = \prod^{sh}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot sh(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot sh(\alpha_1)^{-1} \cdot sh(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $sh^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, \ldots) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot sh(\beta_2) \cdot sh^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, \ldots$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, \ldots) \bullet \beta = (\beta_1, \beta_2, \beta_3, \ldots)$. Then $\beta_1, \beta_2, \ldots$ are special.
As $\prod^{sh}(1, 1, 1\ldots) = 1$, the lemma implies $\beta = \prod^{sh}\vec{\beta}$.

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.

• Lemma: *For* $\vec\alpha = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{sh}\vec\alpha$ *for* $\alpha_1 \cdot sh(\alpha_2) \cdot sh^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec\alpha \bullet \beta = \vec\gamma \quad \text{implies} \quad \prod^{sh}\vec\alpha \cdot \beta = \prod^{sh}\vec\gamma.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec\alpha$ contributes $\alpha_1 \cdot sh(\alpha_2) \cdot \cdots$, whereas $\vec\gamma$ contributes
$\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot sh(\alpha_1)^{-1} \cdot sh(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $sh^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot sh(\beta_2) \cdot sh^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{sh}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{sh}\vec\beta$.
Conversely, assume $\beta = \prod^{sh}\vec\beta$. Then $(1, 1, 1, ...) \bullet \beta$ is defined,

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}} \vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}} \vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}} \vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}} \vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$,

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathsf{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \mathsf{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathsf{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathsf{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha_1)^{-1} \cdot \mathsf{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathsf{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathsf{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathsf{sh}(\beta_2) \cdot \mathsf{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathsf{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathsf{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathsf{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

• <u>Lemma</u>: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• <u>Proposition</u>: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

• At this point, two main questions:

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{sh}\vec{\alpha}$ *for* $\alpha_1 \cdot sh(\alpha_2) \cdot sh^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \text{implies} \quad \prod^{sh}\vec{\alpha} \cdot \beta = \prod^{sh}\vec{\gamma}.$$

▸ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot sh(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot sh(\alpha_1)^{-1} \cdot sh(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $sh^2(\alpha_i)$, that's OK. □

- Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot sh(\beta_2) \cdot sh^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

  ▸ Applies in particular to every positive braid.

▸ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{sh}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{sh}\vec{\beta}$.
Conversely, assume $\beta = \prod^{sh}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

- At this point, two main questions:

  ▸ Can one use the braid shelf and the associated diagram colorings in topology?

• <u>Lemma</u>: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• <u>Proposition</u>: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

• At this point, two main questions:

▶ Can one use the braid shelf and the associated diagram colorings in topology?
⇝ already used to define and investigate the braid ordering

- Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}}\vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}}\vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}}\vec{\gamma}.$$

▸ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

- Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

  - Applies in particular to every positive braid.

▸ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}}\vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

- At this point, two main questions:
  - Can one use the braid shelf and the associated diagram colorings in topology?
    - ⇝ already used to define and investigate the braid ordering
    - ⇝ new applications?

• <u>Lemma</u>: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{\mathrm{sh}} \vec{\alpha}$ *for* $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \mathrm{sh}^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \textit{implies} \quad \prod^{\mathrm{sh}} \vec{\alpha} \cdot \beta = \prod^{\mathrm{sh}} \vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha_1)^{-1} \cdot \mathrm{sh}(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot \mathrm{sh}(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $\mathrm{sh}^2(\alpha_i)$, that's OK. □

• <u>Proposition</u>: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot \mathrm{sh}(\beta_2) \cdot \mathrm{sh}^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{\mathrm{sh}}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{\mathrm{sh}} \vec{\beta}$.
Conversely, assume $\beta = \prod^{\mathrm{sh}} \vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

• At this point, two main questions:

▶ Can one use the braid shelf and the associated diagram colorings in topology?
⇝ already used to define and investigate the braid ordering
⇝ new applications?

▶ Where does this (strange) operation come from?

Plan:

- Describe the free (left) shelf based on a set X

- Describe the free (left) shelf based on a set X ($=$ the most general shelf gen'd by X)

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
(= the shelf generated by X, every shelf generated by X is a quotient of)

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$,*

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
(= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let* $\mathcal{T}_X$ *be the family of all terms built from* X *and* $\triangleright$, *and* $=_{\mathrm{SD}}$ *be the congruence (i.e., compatible equiv. rel.) on* $\mathcal{T}_X$ *generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ *is the free left-shelf based on* X.

▶ Proof: trivial. □

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
(= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far.

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X / {=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X:

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

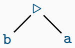
  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X: assuming $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
(= the shelf generated by X, every shelf generated by X is a quotient of)

- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*
$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$
Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.
  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X: assuming $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,



$\mathtt{b} \triangleright \mathtt{a}$

- Describe the free (left) shelf based on a set X (= the most general shelf gen'd by X)
  (= the shelf generated by X, every shelf generated by X is a quotient of)

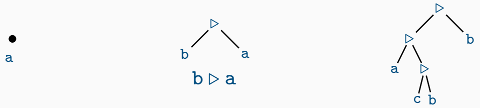
- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X: assuming $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,



$\mathtt{b} \triangleright \mathtt{a}$

$(\mathtt{a} \triangleright (\mathtt{b} \triangleright \mathtt{c})) \triangleright \mathtt{b}$

- Lemma (confluence): *Let* $\rightarrow_{\text{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\text{SD}} T_2$ *holds iff*

- Lemma (confluence): *Let* $\to_{SD}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{SD} T_2$ *holds iff one has* $T_1 \to_{SD} T$ *and* $T_2 \to_{SD} T$ *for some* T.

- Lemma (confluence): *Let* $\rightarrow_{\mathrm{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \rightarrow_{\mathrm{SD}} T$ *and* $T_2 \rightarrow_{\mathrm{SD}} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

- Lemma (confluence): *Let* $\rightarrow_{\mathrm{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \rightarrow_{\mathrm{SD}} T$ *and* $T_2 \rightarrow_{\mathrm{SD}} T$ *for some* T*.*

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\rightarrow_{\mathrm{SD}}$ (clear):
  $T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\rightarrow_{\mathrm{SD}}$-zigzag from T_1 to T_2.

- Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the* semi*-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

  - Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):
    $T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.
    Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T'$ $(T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

- Lemma (confluence): *Let* $\to_{SD}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{SD} T_2$ *holds iff one has* $T_1 \to_{SD} T$ *and* $T_2 \to_{SD} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{SD}$ is the symmetric closure of $\to_{SD}$ (clear):

  $T_1 =_{SD} T_2$ holds iff there is a $\to_{SD}$-zigzag from T_1 to T_2.

  Suffices to show: if $T \to_{SD} T_1$ and $T \to_{SD} T_2$, then $\exists T' \,(T_1 \to_{SD} T'$ and $T_2 \to_{SD} T')$.

  Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,

• Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the* semi*-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

▶ Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T' \, (T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$.

- Lemma (confluence): *Let $\to_{\mathrm{SD}}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ holds iff one has $T_1 \to_{\mathrm{SD}} T$ and $T_2 \to_{\mathrm{SD}} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T'\,(T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

  - $T \to_{\mathrm{SD}} \partial T$ (easy induction),

- Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

▶ Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T'\, (T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \to_{\mathrm{SD}} \partial T$ (easy induction),
- $T \to^1_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial T$ (semi-easy induction),

- Lemma (confluence): *Let $\rightarrow_{SD}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{SD} T_2$ holds iff one has $T_1 \rightarrow_{SD} T$ and $T_2 \rightarrow_{SD} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{SD}$ is the symmetric closure of $\rightarrow_{SD}$ (clear):

$T_1 =_{SD} T_2$ holds iff there is a $\rightarrow_{SD}$-zigzag from T_1 to T_2.

Suffices to show: if $T \rightarrow_{SD} T_1$ and $T \rightarrow_{SD} T_2$, then $\exists T'$ $(T_1 \rightarrow_{SD} T'$ and $T_2 \rightarrow_{SD} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \rightarrow_{SD} \partial T$ (easy induction),
- $T \rightarrow^1_{SD} T'$ implies $T' \rightarrow_{SD} \partial T$ (semi-easy induction),
- $T \rightarrow_{SD} T'$ implies $\partial T \rightarrow_{SD} \partial T'$ (more delicate induction).

- Lemma (confluence): *Let $\rightarrow_{\mathrm{SD}}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ holds iff one has $T_1 \rightarrow_{\mathrm{SD}} T$ and $T_2 \rightarrow_{\mathrm{SD}} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

▶ Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\rightarrow_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\rightarrow_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \rightarrow_{\mathrm{SD}} T_1$ and $T \rightarrow_{\mathrm{SD}} T_2$, then $\exists T' \, (T_1 \rightarrow_{\mathrm{SD}} T'$ and $T_2 \rightarrow_{\mathrm{SD}} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \rightarrow_{\mathrm{SD}} \partial T$ (easy induction),
- $T \rightarrow^1_{\mathrm{SD}} T'$ implies $T' \rightarrow_{\mathrm{SD}} \partial T$ (semi-easy induction),
- $T \rightarrow_{\mathrm{SD}} T'$ implies $\partial T \rightarrow_{\mathrm{SD}} \partial T'$ (more delicate induction).

From there, $T \rightarrow^p_{\mathrm{SD}} T'$ implies $T' \rightarrow_{\mathrm{SD}} \partial^p T$ (easy),

- Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T*.*

"SD-equivalent iff admit a common SD-expansion"

▶ Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T'$ ($T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T'$).

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \to_{\mathrm{SD}} \partial T$ (easy induction),
- $T \to^1_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial T$ (semi-easy induction),
- $T \to_{\mathrm{SD}} T'$ implies $\partial T \to_{\mathrm{SD}} \partial T'$ (more delicate induction).

From there, $T \to^p_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial^p T$ (easy), whence

- Lemma (confluence): *Let $\to_{\mathrm{SD}}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ holds iff one has $T_1 \to_{\mathrm{SD}} T$ and $T_2 \to_{\mathrm{SD}} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

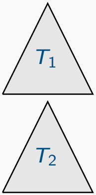
$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T' \,(T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \to_{\mathrm{SD}} \partial T$ (easy induction),
- $T \to^1_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial T$ (semi-easy induction),
- $T \to_{\mathrm{SD}} T'$ implies $\partial T \to_{\mathrm{SD}} \partial T'$ (more delicate induction).

From there, $T \to^p_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial^p T$ (easy), whence
$T \to^p_{\mathrm{SD}} T_1$ and $T \to^q_{\mathrm{SD}} T_2$ implies $T_1 \to_{\mathrm{SD}} \partial^r T$ and $T_2 \to_{\mathrm{SD}} \partial^r T$ for $r \geqslant \max(p, q)$. □



- Lemma (confluence): *Let $\to_{\mathrm{SD}}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ holds iff one has $T_1 \to_{\mathrm{SD}} T$ and $T_2 \to_{\mathrm{SD}} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

- Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T' \,(T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T')$.

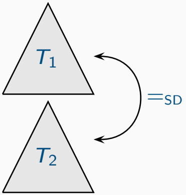
Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,

and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \to_{\mathrm{SD}} \partial T$ (easy induction),
- $T \to^1_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial T$ (semi-easy induction),
- $T \to_{\mathrm{SD}} T'$ implies $\partial T \to_{\mathrm{SD}} \partial T'$ (more delicate induction).

From there, $T \to^p_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial^p T$ (easy), whence

$T \to^p_{\mathrm{SD}} T_1$ and $T \to^q_{\mathrm{SD}} T_2$ implies $T_1 \to_{\mathrm{SD}} \partial^r T$ and $T_2 \to_{\mathrm{SD}} \partial^r T$ for $r \geqslant \max(p, q)$. □



- Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T.

"SD-equivalent iff admit a common SD-expansion"

▶ Proof: $=_{\mathrm{SD}}$ is the symmetric closure of $\to_{\mathrm{SD}}$ (clear):

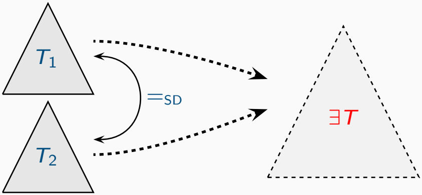
$T_1 =_{\mathrm{SD}} T_2$ holds iff there is a $\to_{\mathrm{SD}}$-zigzag from T_1 to T_2.

Suffices to show: if $T \to_{\mathrm{SD}} T_1$ and $T \to_{\mathrm{SD}} T_2$, then $\exists T'$ ($T_1 \to_{\mathrm{SD}} T'$ and $T_2 \to_{\mathrm{SD}} T'$).

Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then

- $T \to_{\mathrm{SD}} \partial T$ (easy induction),
- $T \to^1_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial T$ (semi-easy induction),
- $T \to_{\mathrm{SD}} T'$ implies $\partial T \to_{\mathrm{SD}} \partial T'$ (more delicate induction).

From there, $T \to^p_{\mathrm{SD}} T'$ implies $T' \to_{\mathrm{SD}} \partial^p T$ (easy), whence
$T \to^p_{\mathrm{SD}} T_1$ and $T \to^q_{\mathrm{SD}} T_2$ implies $T_1 \to_{\mathrm{SD}} \partial^r T$ and $T_2 \to_{\mathrm{SD}} \partial^r T$ for $r \geqslant \max(p, q)$. □



- Lemma (absorption):

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$.

• Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$,

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{SD} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

  - Proof: Induction on T.

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

  - Proof: Induction on T. For $T = x$, direct from the definitions.

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$.

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

  - Proof: Induction on T. For $T = x$, direct from the definitions.
    Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
    Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
    $=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{SD} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
Then $x^{[n+1]} =_{SD} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
$=_{SD} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2
$=_{SD} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD
$=_{SD} (T_1 \triangleright T_2) \triangleright x^{[n]}$ by induction hypothesis for T_1

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.

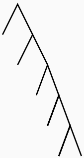
$$\begin{aligned}
\text{Then } x^{[n+1]} &=_{\mathrm{SD}} T_1 \triangleright x^{[n]} && \text{by induction hypothesis for } T_1\\
&=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]}) && \text{by induction hypothesis for } T_2\\
&=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]}) && \text{by applying SD}\\
&=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]} && \text{by induction hypothesis for } T_1\\
&= T \triangleright x^{[n]}. && \square
\end{aligned}$$

- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]}$ by induction hypothesis for T_1
$= T \triangleright x^{[n]}$. □

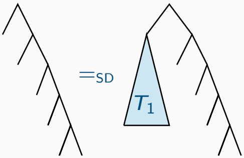


- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]}$ by induction hypothesis for T_1
$= T \triangleright x^{[n]}$. □



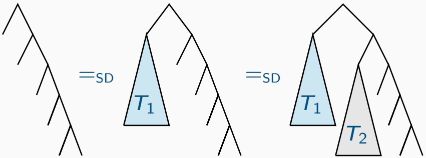
- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{SD} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▸ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n-1 > \mathrm{ht}(T_1)$ and $n-1 > \mathrm{ht}(T_2)$.

Then $x^{[n+1]} =_{SD} T_1 \triangleright x^{[n]}$	by induction hypothesis for T_1
$=_{SD} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$	by induction hypothesis for T_2
$=_{SD} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$	by applying SD
$=_{SD} (T_1 \triangleright T_2) \triangleright x^{[n]}$	by induction hypothesis for T_1
$= T \triangleright x^{[n]}$.	□



- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

  - Proof: Induction on T. For $T = x$, direct from the definitions. Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.

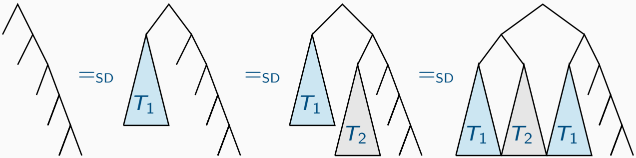
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ — by induction hypothesis for T_1

$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ — by induction hypothesis for T_2

$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ — by applying SD

$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]}$ — by induction hypothesis for T_1

$= T \triangleright x^{[n]}$. □



- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{SD} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions. Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.

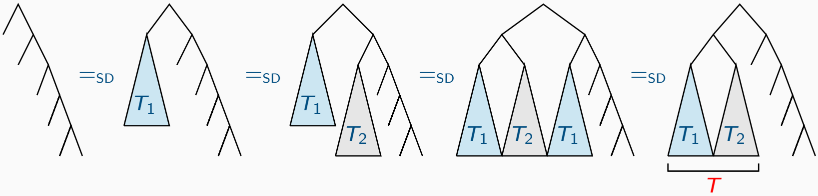
Then $x^{[n+1]} =_{SD} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1

$=_{SD} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2

$=_{SD} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD

$=_{SD} (T_1 \triangleright T_2) \triangleright x^{[n]}$ by induction hypothesis for T_1

$= T \triangleright x^{[n]}$. □



- Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' \, (T' =_{SD} T \triangleright T'')$,

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$*, and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$*.*

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T''\,(T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T',$$

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T',$$

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T''\,(T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

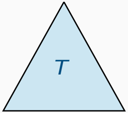
$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

▶ Proof:

• <u>Lemma</u> (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

▶ Proof:



- Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \,(T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

  - Proof:

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T''\,(T' =_{\mathrm{SD}} T \rhd T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

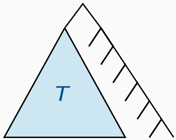
$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

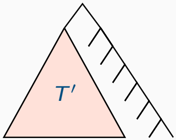
- Proof:

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \,(T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \qquad T =_{\mathrm{SD}} T', \qquad T' \sqsubset^*_{\mathrm{SD}} T.$$

- Proof:

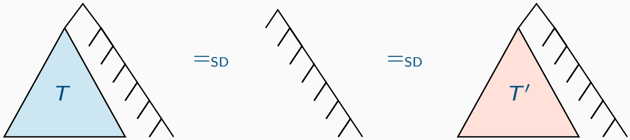




• Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

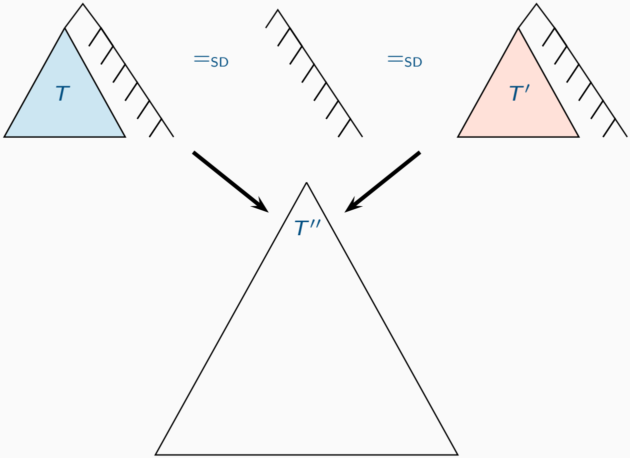
- Proof:



- Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' \, (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

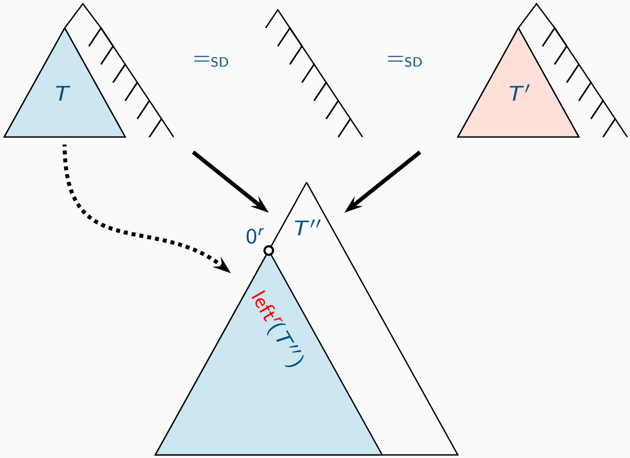
  - Proof:



• Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

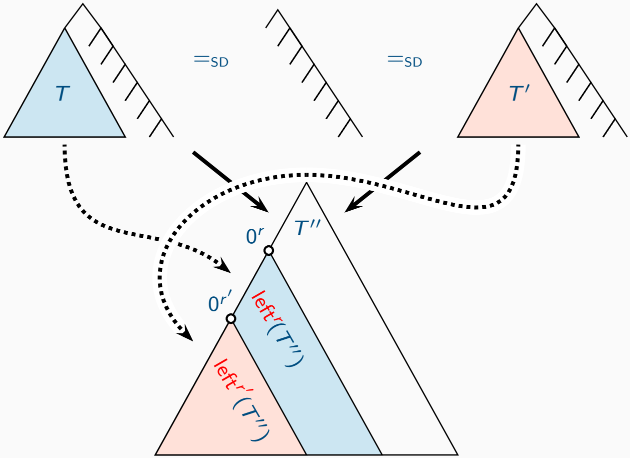
▶ Proof:



• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \quad T =_{\mathrm{SD}} T', \quad T' \sqsubset^*_{\mathrm{SD}} T.$$

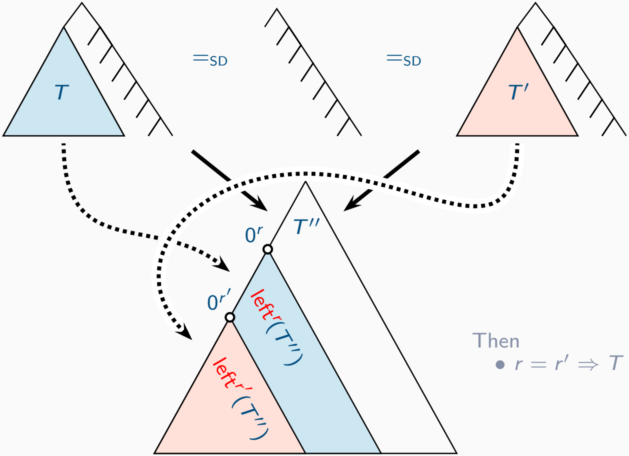
▶ Proof:



• <u>Lemma</u> (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T''\,(T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \qquad T =_{SD} T', \qquad T' \sqsubset^*_{SD} T.$$

▶ Proof:



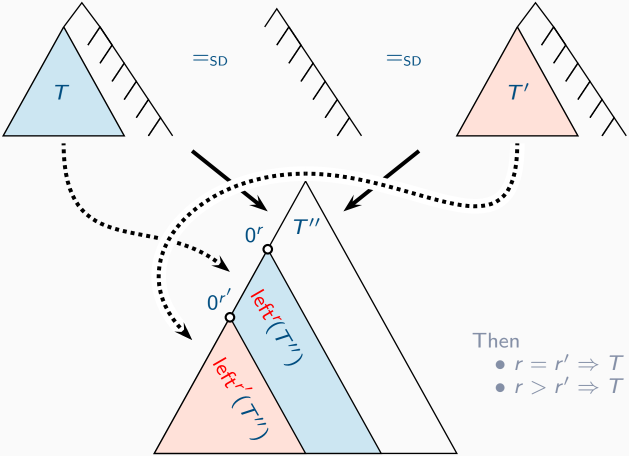
Then

• $r = r' \Rightarrow T =_{SD} T'$

• Lemma (comparison I): *Write* $T \sqsubset_{\mathrm{SD}} T'$ *for* $\exists T'' \, (T' =_{\mathrm{SD}} T \triangleright T'')$, *and* $\sqsubset^*_{\mathrm{SD}}$ *for the transitive closure of* $\sqsubset_{\mathrm{SD}}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{\mathrm{SD}} T', \qquad T =_{\mathrm{SD}} T', \qquad T' \sqsubset^*_{\mathrm{SD}} T.$$

▶ Proof:

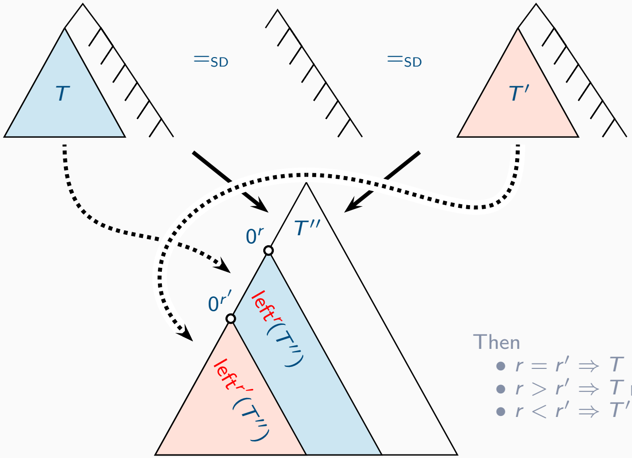


Then
- $r = r' \Rightarrow T =_{\mathrm{SD}} T'$
- $r > r' \Rightarrow T \sqsubset^*_{\mathrm{SD}} T'$

- Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' \, (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

- Proof:



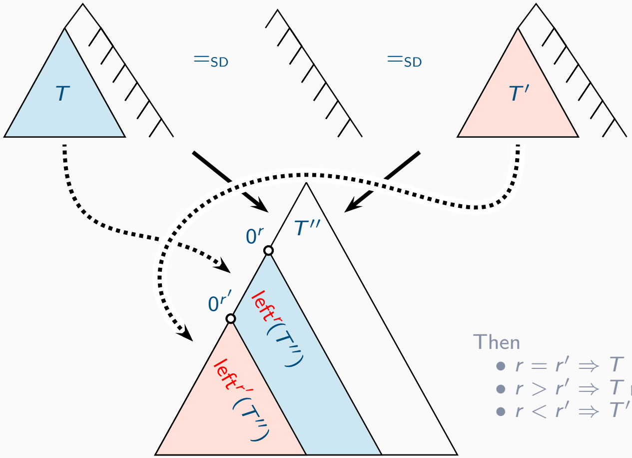
Then

- $r = r' \Rightarrow T =_{SD} T'$
- $r > r' \Rightarrow T \sqsubset^*_{SD} T'$
- $r < r' \Rightarrow T' \sqsubset^*_{SD} T$

- Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' \, (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

▶ Proof:



Then
- $r = r' \Rightarrow T =_{SD} T'$
- $r > r' \Rightarrow T \sqsubset^*_{SD} T'$
- $r < r' \Rightarrow T' \sqsubset^*_{SD} T$

□

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$

transitive closure of $\sqsubset$ = iterated left divisibility relation

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$
transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S.

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$
transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'.

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$
transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S;

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S;

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$
transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. $\square$

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$
transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g.

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

↑ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$".

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$

• Lemma (comparison II): *If $(S, \triangleright)$ is a monogenerated left-shelf, any two distinct elements of S are $\sqsubset^*$-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If $(S, \triangleright)$ is a monogenerated left-shelf and $\sqsubset$ has no cycle, then $(S, \triangleright)$ is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$,

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$,

whence $T(g) \sqsubset^* T'(g)$

• Lemma (comparison II): *If $(S, \triangleright)$ is a monogenerated left-shelf, any two distinct elements of S are $\sqsubset^*$-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If $(S, \triangleright)$ is a monogenerated left-shelf and $\sqsubset$ has no cycle, then $(S, \triangleright)$ is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$,

whence $T(g) \sqsubset^* T'(g)$ or $T'(g) \sqsubset^* T(g)$.

• Lemma (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

↑ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• Proposition (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$, whence $T(g) \sqsubset^* T'(g)$ or $T'(g) \sqsubset^* T(g)$. As $\sqsubset$ has no cycle in S, both imply $T(g) \neq T'(g)$. □

- <u>Definition</u>: For α a binary address (= finite sequence of 0s and 1s),

• Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α".

• Definition: For α a binary address $(=$ finite sequence of 0s and 1s$)$, let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

• Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

• Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

• Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathsf{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

• Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathsf{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathsf{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathsf{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - ▶ Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathsf{sh}_1(\chi_{T_2}) \cdot \mathsf{SD}_\emptyset \cdot \mathsf{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathsf{SD}}$:

  $\mathsf{SD}_{11\alpha}\mathsf{SD}_\alpha = \mathsf{SD}_\alpha\mathsf{SD}_{11\alpha}$,

• Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

• Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

• Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.

▶ Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

• Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\ldots1}$s remain.

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathsf{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathsf{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathsf{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \qquad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathsf{sh}_1(\chi_{T_2}) \cdot \mathsf{SD}_\emptyset \cdot \mathsf{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathsf{SD}}$:

$$\mathsf{SD}_{11\alpha}\mathsf{SD}_\alpha = \mathsf{SD}_\alpha\mathsf{SD}_{11\alpha}, \quad \mathsf{SD}_{1\alpha}\mathsf{SD}_\alpha\mathsf{SD}_{1\alpha}\mathsf{SD}_{0\alpha} = \mathsf{SD}_\alpha\mathsf{SD}_{1\alpha}\mathsf{SD}_\alpha, \text{ etc.} \qquad (**)$$

  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathsf{SD}_{11\ldots1}$s remain.
  - Write σ_{i+1} for the image of $\mathsf{SD}_{11\ldots1}$, i times 1. Then $(**)$ becomes

$\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j-i| \geqslant 2$,

• Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

• Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

• Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  ▸ Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

• Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

  ▸ When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\ldots1}$s remain.
  ▸ Write σ_{i+1} for the image of $\mathrm{SD}_{11\ldots1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

• Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

• Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

• Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

• Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\ldots1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11\ldots1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \qquad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11...1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11...1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \qquad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).
  - If ϕ maps T to T', then $\mathrm{sh}_0(\phi)$ maps $T \triangleright x^{[n]}$ to $T' \triangleright x^{[n]}$,

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\dots1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11\dots1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).
  - If ϕ maps T to T', then $\mathrm{sh}_0(\phi)$ maps $T \triangleright x^{[n]}$ to $T' \triangleright x^{[n]}$, so collapsing all $\mathrm{sh}_0(\phi)$ must give an SD-operation on the quotient, i.e., on B_∞.

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

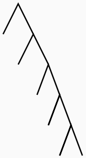
  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\dots1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11\dots1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

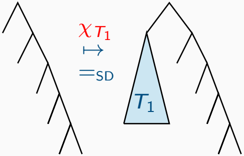
  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).
  - If ϕ maps T to T', then $\mathrm{sh}_0(\phi)$ maps $T \triangleright x^{[n]}$ to $T' \triangleright x^{[n]}$, so collapsing all $\mathrm{sh}_0(\phi)$ must give an SD-operation on the quotient, i.e., on B_∞.
  - Its definition is the projection of $(*)$, i.e.,

$$a \triangleright b := a \cdot \mathrm{sh}(b) \cdot \sigma_1 \cdot \mathrm{sh}(a)^{-1}.$$

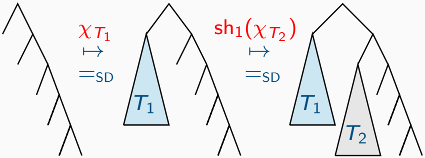
- The "magic rule" revisited:



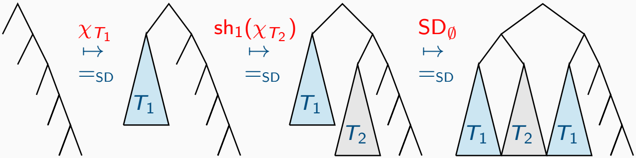
- The "magic rule" revisited:



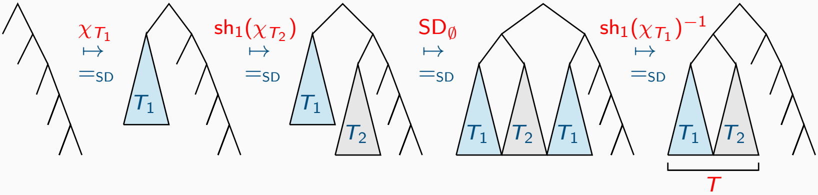
- The "magic rule" revisited:



- The "magic rule" revisited:



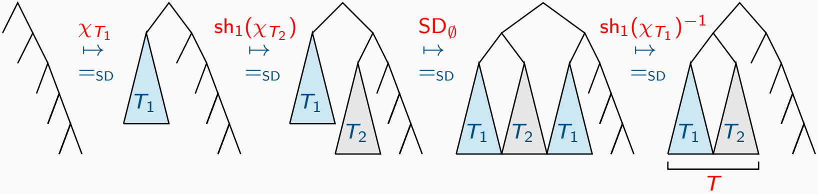
- The "magic rule" revisited:



$$\text{whence } \chi_{T_1 \triangleright T_2} = \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1}^{-1}),$$

which projects to the braid operation.

- The "magic rule" revisited:

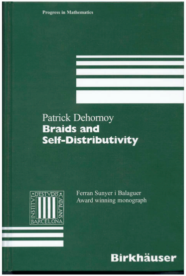


$$\text{whence } \chi_{T_1 \triangleright T_2} = \chi_{T_1} \cdot sh_1(\chi_{T_2}) \cdot SD_\emptyset \cdot sh_1(\chi_{T_1}^{-1}),$$

which projects to the braid operation.

.../...

- See more in [P.D., Braids and selfdistributivity, PM192, Birkhaüser (1999)]



Plan:

- From the very beginning, Set Theory is a theory of infinity.

- From the very beginning, Set Theory is a theory of infinity.

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities,*

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$,

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- Question: For which α (necessarily $\geqslant 1$) does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- Question: For which α (necessarily $\geqslant 1$) does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?
  - Conjecture (Continuum Hypothesis, Cantor, 1879):

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- Question: For which α (necessarily $\geqslant 1$) does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?
  - Conjecture (Continuum Hypothesis, Cantor, 1879): $\mathrm{card}(\mathbb{R}) = \aleph_1$.

- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- Question: For which α (necessarily $\geqslant 1$) does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?
  - Conjecture (Continuum Hypothesis, Cantor, 1879): $\mathrm{card}(\mathbb{R}) = \aleph_1$.
  - Equivalently: every uncountable set of reals has the cardinality of $\mathbb{R}$.

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...)

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...)
  and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*"

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future...)*

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future...)*
  - First question:

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future...)*
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...)
  and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future...)*
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" *(but this may change in the future...)*
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,*

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑ negation of

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑
negation of

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* ¬CH *cannot be proved from* ZF.

↑ negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - ▶ Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - ▶ First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑ negation of

- Theorem (Cohen, 1963): *Unless ZF is contradictory, CH cannot be proved from ZF.*

▶ Method of proof: Investigate models of ZF = abstract structures satisfying the axioms of ZF

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑ negation of

- Theorem (Cohen, 1963): *Unless ZF is contradictory, CH cannot be proved from ZF.*

  - Method of proof: Investigate models of ZF = abstract structures satisfying the axioms of ZF (≈ investigate abstract groups or fields).

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑ negation of

- Theorem (Cohen, 1963): *Unless ZF is contradictory, CH cannot be proved from ZF.*

- Method of proof: Investigate models of ZF = abstract structures satisfying the axioms of ZF (≈ investigate abstract groups or fields).
- For Gödel: every model has a submodel that satisfies AC.

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

- Theorem (Gödel, 1938): *Unless ZF is contradictory, ¬CH cannot be proved from ZF.*

↑ negation of

- Theorem (Cohen, 1963): *Unless ZF is contradictory, CH cannot be proved from ZF.*

  - Method of proof: Investigate models of ZF = abstract structures satisfying the axioms of ZF (≈ investigate abstract groups or fields).
  - For Gödel: every model has a submodel that satisfies AC.
  - For Cohen: every model has an extension that satisfies ¬AC.

- <u>Conclusion</u>:

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea...

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea...

- Which new axioms?

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)

    Example: CH may be taken as an additional axiom, but not a good idea...

- Which new axioms?

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea...

- Which new axioms?

- From 1930's, axioms of large cardinal:

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea...

- Which new axioms?

- From 1930's, axioms of large cardinal:

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)

    Example: CH may be taken as an additional axiom, but not a goo ea...

- Which new axioms?

- From 1930's, axioms of large cardinal:

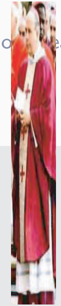
- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea...

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}}$$

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea...



- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

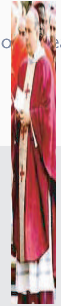
$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
    - Discover further properties of sets, and adopt an extended list of axioms!
    - How to recognize that an axiom is true? (What does this mean?)
      Example: CH may be taken as an additional axiom, but not a goo[illegible]ea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
    - various solutions to the equation
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
    - set theory (as opposed to number theory) begins
      when "there exists an infinite set" is in the base axioms;

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…



- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo[...]ea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals,

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea...

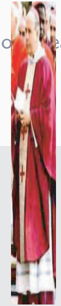
- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals,

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…



- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals,

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- Theorem(s) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom,*
PD

- <u>Conclusion</u>: ZF is incomplete (<u>not</u>: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea...

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- <u>Theorem</u>(<u>s</u>) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom, PD ("projective determinacy",*

- <u>Conclusion</u>: ZF is incomplete (<u>not</u>: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a good idea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- <u>Theorem</u>(<u>s</u>) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom,* PD *("projective determinacy", aka "there exists infinitely many Woodin cardinals"),*

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- Theorem(s) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom, PD ("projective determinacy", aka "there exists infinitely many Woodin cardinals"), provides a heuristically complete description of finite and countable sets.*

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…

- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins
    when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- Theorem(s) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom, PD ("projective determinacy", aka "there exists infinitely many Woodin cardinals"), provides a heuristically complete description of finite and countable sets.*

- New consensus: ZF+PD is, from now on, the reference system for set theory.

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not bject. preserving all $\in$-definable notions)

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑

    an elementary embedding of X

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    $\uparrow$ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
    Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
  Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$", and $<$ is definable from $\in$.

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
    Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
    and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
    Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$", and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$". Hence necessarily $j(0) = 0$.

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.

  ▶ Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
  Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
  and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".
  Hence necessarily $j(0) = 0$. Now 1 says "I am the smallest after 0":

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
    Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
    and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".
    Hence necessarily $j(0) = 0$. Now 1 says "I am the smallest after 0":
    By the same argument $j(1) = 1$,

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.

  ▶ Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
  Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
  and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".
  Hence necessarily $j(0) = 0$. Now 1 says "I am the smallest after 0":
  By the same argument $j(1) = 1$, etc. So j is the identity. □

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.

  ▶ Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
  Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
  and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".
  Hence necessarily $j(0) = 0$. Now 1 says "I am the smallest after 0":
  By the same argument $j(1) = 1$, etc. So j is the identity. □

  - A super-infinite set must be so large that it contains undefinable elements (since all definable elements must be fixed).

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal,*

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*

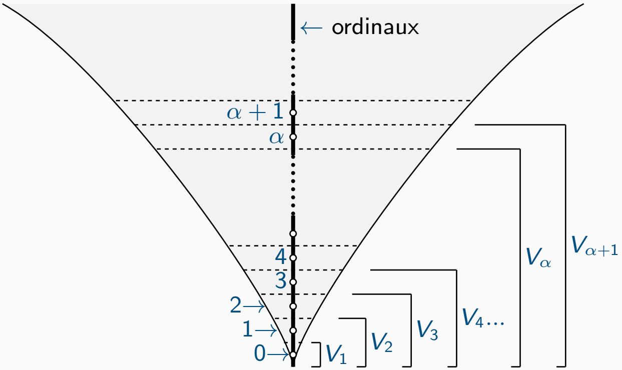
  $V_0 := \emptyset$,

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*

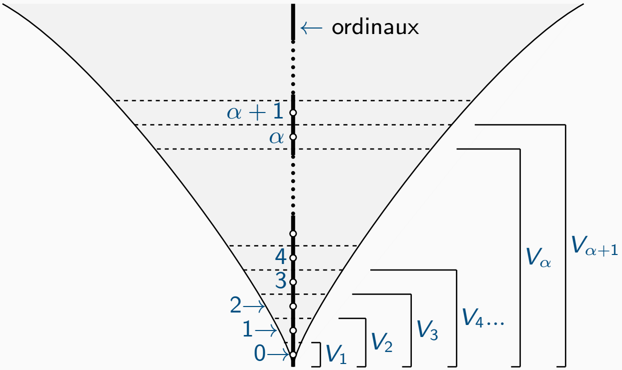
$$V_0 := \emptyset, \qquad V_{\alpha+1} := \mathfrak{P}(V_\alpha),$$

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
  $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by* $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*

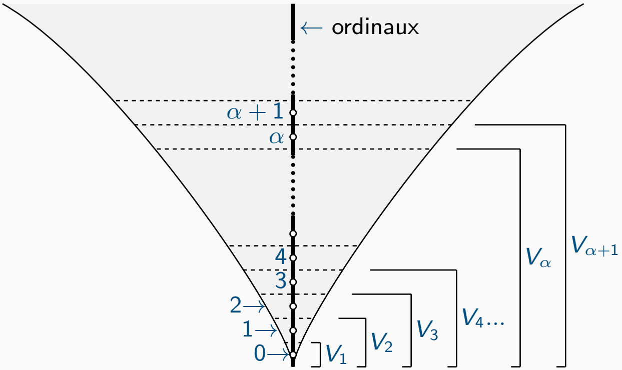


- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
$$V_0 := \emptyset, \quad V_{\alpha+1} := \mathfrak{P}(V_\alpha), \quad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \text{ for } \lambda \text{ limit.}$$



- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,

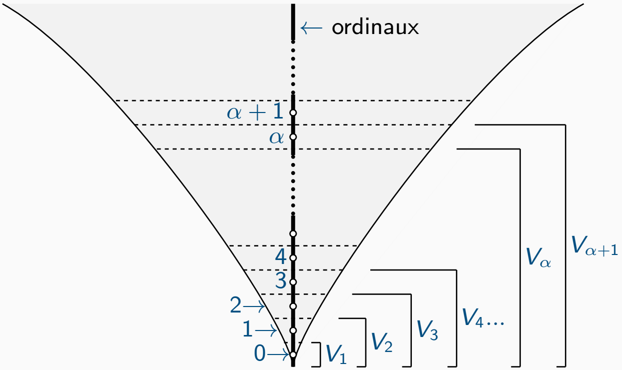
- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
  $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*



- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$

• Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*

$$V_0 := \emptyset, \quad V_{\alpha+1} := \mathfrak{P}(V_\alpha), \quad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \text{ for } \lambda \text{ limit.}$$

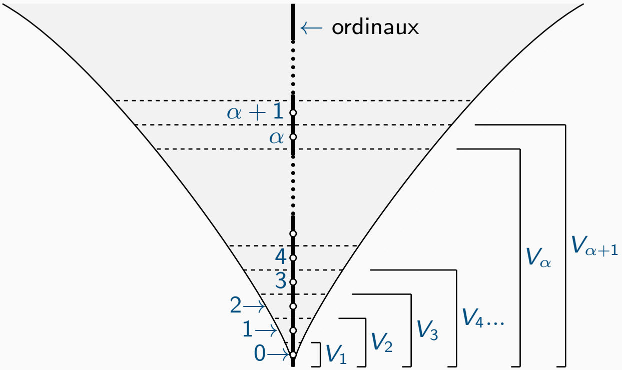


• Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,

then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ *and* $f \cap V_\alpha^2$ *belongs to* V_λ *for every* $\alpha < \lambda$.

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*

  $V_0 := \emptyset, \quad V_{\alpha+1} := \mathfrak{P}(V_\alpha), \quad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*

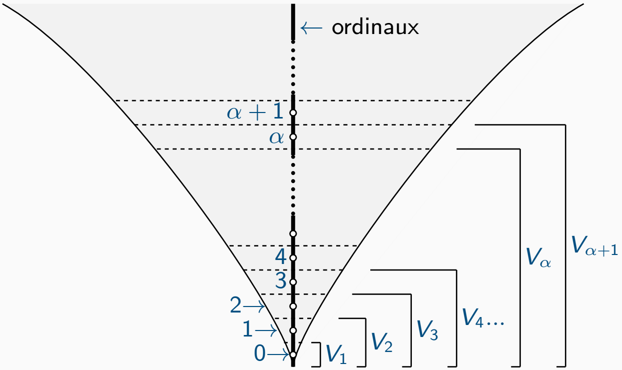


- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,

  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ *and* $f \cap V_\alpha^2$ *belongs to* V_λ *for every* $\alpha < \lambda$.

  - Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$;

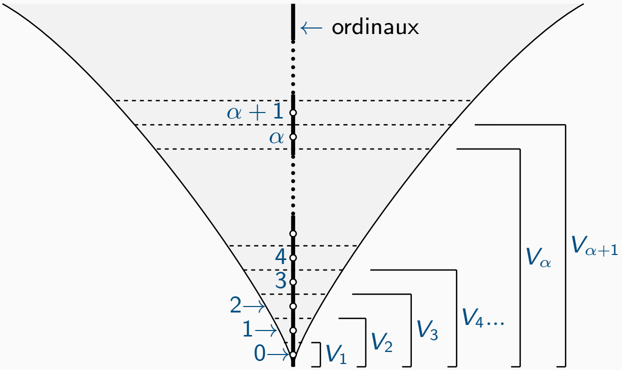
- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
  $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*



- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ *and* $f \cap V_\alpha^2$ *belongs to* V_λ *for every* $\alpha < \lambda$.
  - Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2,

- Fact: *There is a canonical filtration of sets by the sets V_α, α an ordinal, def'd by*

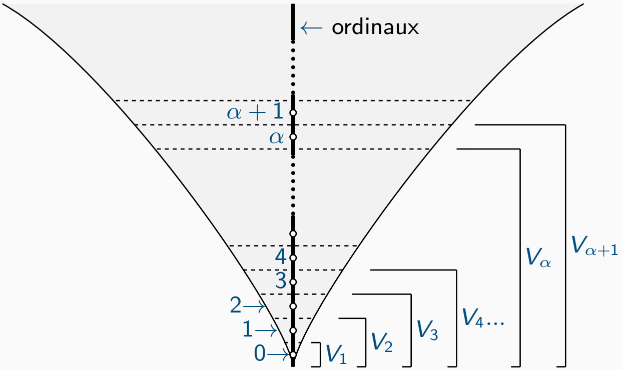
$$V_0 := \emptyset, \qquad V_{\alpha+1} := \mathfrak{P}(V_\alpha), \qquad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \text{ for } \lambda \text{ limit.}$$



- Fact: *If λ is a limit ordinal and $f : V_\lambda \to V_\lambda$,*
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ and $f \cap V_\alpha^2$ belongs to V_λ for every $\alpha < \lambda$.

  ▶ Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2, hence in $V_{\alpha+2}$,

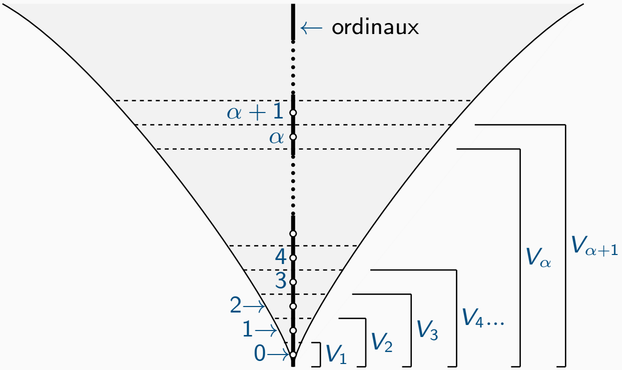
- Fact: *There is a canonical filtration of sets by the sets V_α, α an ordinal, def'd by*
  $V_0 := \emptyset$, $\quad V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $\quad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for λ limit.*



- Fact: *If λ is a limit ordinal and $f : V_\lambda \to V_\lambda$,*
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ and $f \cap V_\alpha^2$ belongs to V_λ for every $\alpha < \lambda$.

  ▶ Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2, hence in $V_{\alpha+2}$, hence it belongs to $V_{\alpha+3}$,

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
  $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*



- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ *and* $f \cap V_\alpha^2$ *belongs to* V_λ *for every* $\alpha < \lambda$.

  ▶ Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2, hence in $V_{\alpha+2}$, hence it belongs to $V_{\alpha+3}$, hence to V_λ. □

- Definition: *A Laver cardinal is a cardinal* λ *s.t. the set* V_λ *is "super-infinite",*

- Definition: *A Laver cardinal is a cardinal* λ *s.t. the set* V_λ *is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ *to itself.*

- <u>Definition</u>: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- <u>Fact</u>: *If there exists a super-infinite set, there exists a super-infinite set V_λ*

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

- Definition: *A* *Laver cardinal* *is a cardinal* λ *s.t. the set* V_λ *is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ *to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set* V_λ
  (hence a Laver cardinal).

- Fact: *Assume* $j : V_\lambda \to V_\lambda$ *witnesses that* λ *is a Laver cardinal.*
  - *The map* j *sends every ordinal* α *to an ordinal* $\geqslant \alpha$*.*

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*

- Definition: *A* *Laver cardinal* *is a cardinal* λ *s.t. the set* V_λ *is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ *to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set* V_λ
  (hence a Laver cardinal).

- Fact: *Assume* $j : V_\lambda \to V_\lambda$ *witnesses that* λ *is a Laver cardinal.*
  - *The map* j *sends every ordinal* α *to an ordinal* $\geqslant \alpha$.
  - *There exists an ordinal* α *satisfying* $j(\alpha) > \alpha$.
  - *There exists a smallest ordinal* κ *satisfying* $j(\kappa) > \kappa$:

- Definition: *A Laver cardinal is a cardinal* λ *s.t. the set* V_λ *is "super-infinite", i.e., there exists a non-surjective elementary embedding from* V_λ *to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set* V_λ *(hence a Laver cardinal).*

- Fact: *Assume* $j : V_\lambda \to V_\lambda$ *witnesses that* λ *is a Laver cardinal.*
  - *The map* j *sends every ordinal* α *to an ordinal* $\geqslant \alpha$.
  - *There exists an ordinal* α *satisfying* $j(\alpha) > \alpha$.
  - *There exists a smallest ordinal* κ *satisfying* $j(\kappa) > \kappa$*: the "critical ordinal" of* j.

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

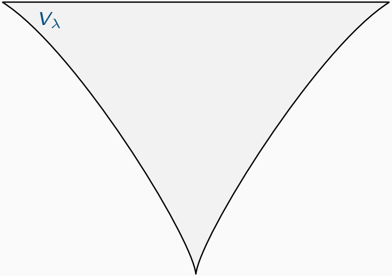
- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \mathrm{sup}_n j^n(\mathrm{crit(j)})$.*

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ to itself.

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ*
  (hence a Laver cardinal).

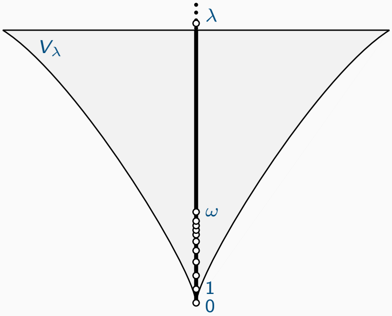
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- <u>Definition</u>: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ to itself.

- <u>Fact</u>: *If there exists a super-infinite set, there exists a super-infinite set V_λ*
  (hence a Laver cardinal).

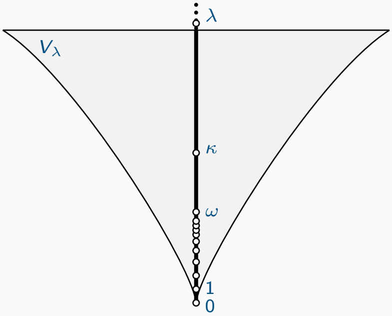
- <u>Fact</u>: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

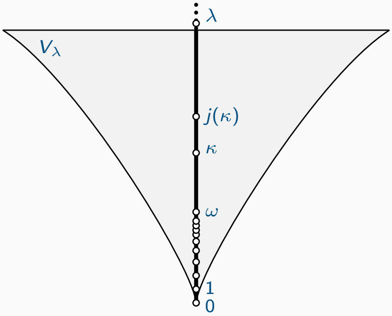
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

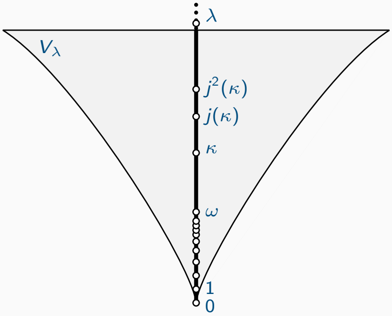
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite",* *i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ* *(hence a Laver cardinal).*

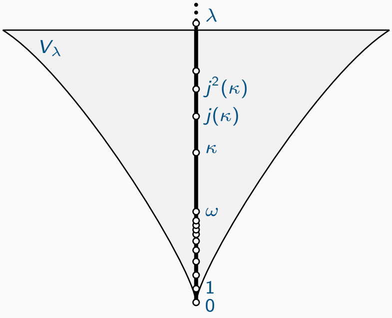
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

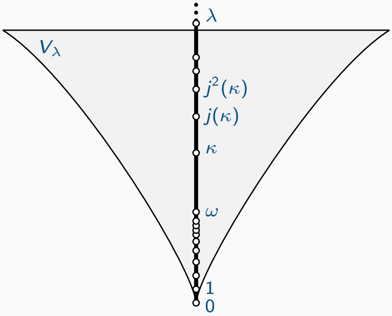
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

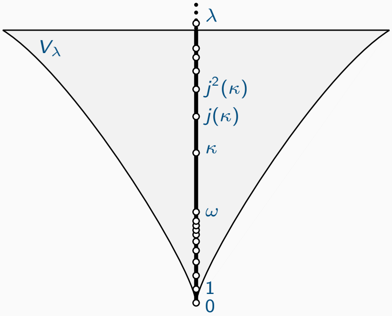
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\text{crit}(j))$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

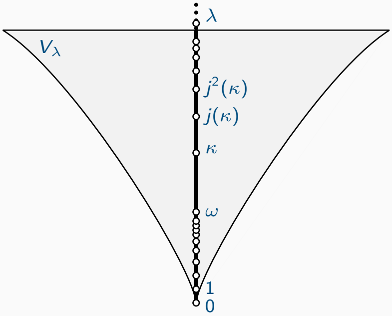
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit}(j))$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

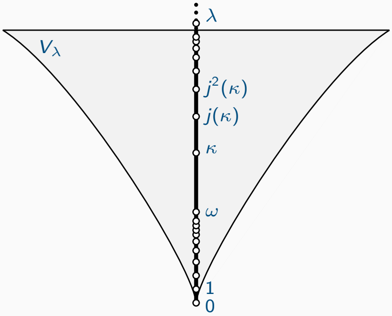
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

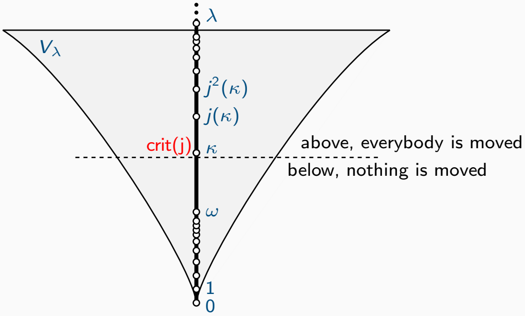
- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- Definition: *A* *Laver cardinal* *is a cardinal* λ *s.t. the set* V_λ *is "super-infinite",*
  i.e., there exists a non-surjective elementary embedding from V_λ *to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set* V_λ
  (hence a Laver cardinal).

- Fact: *Assume* $j : V_\lambda \to V_\lambda$ *witnesses that* λ *is a Laver cardinal.*
  - *The map* j *sends every ordinal* α *to an ordinal* $\geqslant \alpha$.
  - *There exists an ordinal* α *satisfying* $j(\alpha) > \alpha$.
  - *There exists a smallest ordinal* κ *satisfying* $j(\kappa) > \kappa$*: the "critical ordinal" of* j.
  - *One necessarily has* $\lambda = \mathrm{sup}_n j^n(\mathrm{crit(j)})$.



- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*
  - Proof: The sets $j \cap V_\alpha^2$ belong to V_λ,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial ($=$ non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*
  - Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i,j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial $(=$ non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i, j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$, i.e., $i[j[k]] = i[j][i[k]]$:

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial $(=$ non-surjective$)$ elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. $\square$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▸ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j),$$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial $(=$ non-surjective$)$ elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial $(=$ non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial ($=$ non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

   ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
   "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
   "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
   i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

   ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. $\square$

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i,j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa)$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i, j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▸ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▸ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$ implies $\forall \alpha{<}j(\kappa)\, (j[j](\alpha) = \alpha)$,

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$ implies $\forall \alpha{<}j(\kappa)\, (j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall\alpha{<}\kappa\,(j(\alpha) = \alpha)$ implies $\forall\alpha{<}j(\kappa)\,(j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$.

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i, j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall\alpha{<}\kappa\,(j(\alpha) = \alpha)$ implies $\forall\alpha{<}j(\kappa)\,(j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$. We deduce $\mathrm{crit}(j[j]) = j(\kappa)$

- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is

$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i, j) \mapsto i[j]$ *is a binary operation on* E_λ, *and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:

$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$ implies $\forall \alpha{<}j(\kappa)\, (j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$. We deduce $\mathrm{crit}(j[j]) = j(\kappa) > \kappa$. □

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑ closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑ closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▸ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$.

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$.

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) = \mathrm{i}(\mathrm{crit}(\mathrm{i}_1)) \in \mathrm{Im}(\mathrm{i})$,

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$.

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. □

▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. □

▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,

▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself,
then the iterates of j make a left-shelf Iter(j).

↑
closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. $\square$

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) = \mathrm{i}(\mathrm{crit}(\mathrm{i}_1)) \in \mathrm{Im}(\mathrm{i})$, whereas $\mathrm{crit}(\mathrm{i}) \notin \mathrm{Im}(\mathrm{i})$. Hence $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) \neq \mathrm{crit}(\mathrm{i})$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑ closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) = \mathrm{i}(\mathrm{crit}(\mathrm{i}_1)) \in \mathrm{Im}(\mathrm{i})$, whereas $\mathrm{crit}(\mathrm{i}) \notin \mathrm{Im}(\mathrm{i})$. Hence $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) \neq \mathrm{crit}(\mathrm{i})$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD (because both $=_{\mathrm{SD}}$ and $\sqsubset^*_{\mathrm{SD}}$ are semi-decidable).

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD (because both $=_{\mathrm{SD}}$ and $\sqsubset^*_{\mathrm{SD}}$ are semi-decidable).

but all this under the (unprovable) assumption that a Laver cardinal exists.

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1]\cdots[i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{SD}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD (because both $=_{SD}$ and $\sqsubset^*_{SD}$ are semi-decidable).

but all this under the (unprovable) assumption that a Laver cardinal exists.

⇝ motivation for finding another proof/another realization...

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then* Iter(j) *is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(i[i_1]) = i(\mathrm{crit}(i_1)) \in \mathrm{Im}(i)$, whereas $\mathrm{crit}(i) \notin \mathrm{Im}(i)$. Hence $\mathrm{crit}(i[i_1]) \neq \mathrm{crit}(i)$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD (because both $=_{\mathrm{SD}}$ and $\sqsubset^*_{\mathrm{SD}}$ are semi-decidable).

but all this under the (unprovable) assumption that a Laver cardinal exists.

⇝ motivation for finding another proof/another realization...

↑

the braid realization (1992)

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,
  $\mathrm{crit}_\mathrm{n}(\mathrm{j})$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit}(\mathrm{i}) \mid \mathrm{i} \in \mathrm{Iter}(\mathrm{j})\}$.

- Notation: ("left powers") $j_{[p]} := j[j][j]\ldots[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  - One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  - One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself.*

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit}_n(j)$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$.

  - One can show $\mathrm{crit}_0(j) = \mathrm{crit}(j)$, $\mathrm{crit}_1(j) = \mathrm{crit}(j[j])$, $\mathrm{crit}_2(j) = \mathrm{crit}(j[j][j][j])$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i′ agree up to γ") if*

$$\forall x{\in}V_\gamma\,(i(x){\cap}V_\gamma = i'(x){\cap}V_\gamma).$$

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit}_n(j)$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$.

  - One can show $\mathrm{crit}_0(j) = \mathrm{crit}(j)$, $\mathrm{crit}_1(j) = \mathrm{crit}(j[j])$, $\mathrm{crit}_2(j) = \mathrm{crit}(j[j][j][j])$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x{\in}V_\gamma\ (i(x){\cap}V_\gamma = i'(x){\cap}V_\gamma).$$

Then $\equiv_{\mathrm{crit}_n(j)}$ is a congruence on Iter(j),

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  - One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i′ agree up to γ") if*

$$\forall x{\in}V_\gamma\,(i(x){\cap}V_\gamma = i'(x){\cap}V_\gamma).$$

Then $\equiv_{\mathrm{crit_n(j)}}$ is a congruence on Iter(j), it has 2^n classes,

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit}_n(j)$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$.

  - One can show $\mathrm{crit}_0(j) = \mathrm{crit}(j)$, $\mathrm{crit}_1(j) = \mathrm{crit}(j[j])$, $\mathrm{crit}_2(j) = \mathrm{crit}(j[j][j][j])$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x \in V_\gamma \, (i(x) \cap V_\gamma = i'(x) \cap V_\gamma).$$

Then $\equiv_{\mathrm{crit}_n(j)}$ is a congruence on Iter(j), it has 2^n classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$,

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  - One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x{\in}V_\gamma\ (i(x){\cap}V_\gamma = i'(x){\cap}V_\gamma).$$

Then $\equiv_{\mathrm{crit_n(j)}}$ is a congruence on Iter(j), it has 2^n classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  ▶ One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x \in V_\gamma\ (i(x) \cap V_\gamma = i'(x) \cap V_\gamma).$$

Then $\equiv_{\mathrm{crit_n(j)}}$ is a congruence on Iter(j), it has 2^n classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

▶ Proof: (Difficult...) Starts from $j \equiv_{\mathrm{crit(i)}} i[j]$ and similar.
Uses in particular $\mathrm{crit(j_{[m]})} = \mathrm{crit_n(j)}$ with n maximal s.t. 2^n divides m. □

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
  satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$ satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n-1\}$) satisfying $p = 1_{[p]} \bmod 2^n$ for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]} \bmod 2^n$ for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n.

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
  satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]}$ mod 2^n for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- <u>Corollary</u>: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n*.*
  - Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
  satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n - 1\}$) satisfying $p = 1_{[p]} \bmod 2^n$ for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n*.*
  - Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.

  The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]} \bmod 2^n$ for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- <u>Corollary</u>: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n*.*
  - ▶ Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
  satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]} \bmod 2^n$ for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n.
  - ▶ Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
    The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
    By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.
    Then $j_{[2^n]} \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$ implies $j_{[2^n+1]} \equiv_{\mathrm{crit_n(j)}} j$,

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$ satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]}$ mod 2^n for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n.

  ▶ Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
  The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
  By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.
  Then $j_{[2^n]} \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$ implies $j_{[2^n+1]} \equiv_{\mathrm{crit_n(j)}} j$, whence $2^n \triangleright 1 = 1$ in the quotient. □

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$ satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]}$ mod 2^n for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n.
  - ▶ Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
    The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
    By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.
    Then $j_{[2^n]} \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$ implies $j_{[2^n+1]} \equiv_{\mathrm{crit_n(j)}} j$, whence $2^n \triangleright 1 = 1$ in the quotient. □
  - ▶ A set-theoretic realization of A_n as a quotient of the (free) left-shelf Iter(j).

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

  $$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_{\mathrm{n}}(\mathrm{j}); \qquad (*)$$

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_{\mathrm{n}}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_{\mathrm{n}}(\mathrm{j}). \qquad (**)$$

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_{\mathrm{n}}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_{\mathrm{n}}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_{\mathrm{n}}(\mathrm{j})$

- Lemma: *For every* j *in* E_λ*, every term* $t(x)$*, and every* n*,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_\mathrm{n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit}_\mathrm{n}(\mathrm{j})} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n,

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)^{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)^{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$,

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit_n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit_n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit_n}(\mathrm{j})} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

  For $(**)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) = \mathrm{crit_n}(\mathrm{j})$ is the conjunction
    of $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ and $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \not\geqslant \mathrm{crit_{n+1}}(\mathrm{j})$,

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$:

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

  For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}*

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

    For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. ☐

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit_n(j)}$.*

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- <u>Proposition</u> ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

  ▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
  i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

  For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
  of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
  of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. ☐

- <u>Proposition</u> ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
  the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
  iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

  ▶ Proof: Apply the lemma to the term $x_{[p]}$.
  As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$
$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction
of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit_n(j)}$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m(j)} = \mathrm{crit(j_{[2^m]})}$, the embedding $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit(j_{[p]}[j_{[2^m]}])}$,

- <u>Lemma</u>: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction
of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- <u>Proposition</u> ("dictionary"): *For* $m \leqslant n$ *and* $p \leqslant 2^n$,
the period of p *jumps from* 2^m *to* 2^{m+1} *between* A_n *and* A_{n+1}
iff $j_{[p]}$ *maps* $\mathrm{crit_m(j)}$ *to* $\mathrm{crit_n(j)}$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m(j)} = \mathrm{crit(j_{[2^m]})}$, the embedding $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit(j_{[p]}[j_{[2^m]}])}$,
so the RHT is $\mathrm{crit(j_{[p]}[j_{[2^m]}])} = \mathrm{crit_n(j)}$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit}_n(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$.

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction
of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For* $m \leqslant n$ *and* $p \leqslant 2^n$,
the period of p *jumps from* 2^m *to* 2^{m+1} *between* A_n *and* A_{n+1}
iff $j_{[p]}$ *maps* $\mathrm{crit_m(j)}$ *to* $\mathrm{crit_n(j)}$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m(j)} = \mathrm{crit(j_{[2^m]})}$, the embedding $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit(j_{[p]}[j_{[2^m]}])}$,
so the RHT is $\mathrm{crit(j_{[p]}[j_{[2^m]}])} = \mathrm{crit_n(j)}$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(\mathrm{t(j)}) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(\mathrm{t(j)}) = \mathrm{crit_n}(j)$ is the conjunction
of $\mathrm{crit}(\mathrm{t(j)}) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(\mathrm{t(j)}) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit_n}(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.
On the other hand, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$,

- Lemma: *For every* j *in* E_λ*, every term* $t(x)$*, and every* n*,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_\mathrm{n}(\mathrm{j}). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit}_\mathrm{n}(\mathrm{j})} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) = \mathrm{crit}_\mathrm{n}(\mathrm{j})$ is the conjunction
of $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ and $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \not\geqslant \mathrm{crit}_{\mathrm{n+1}}(\mathrm{j})$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For* $m \leqslant n$ *and* $p \leqslant 2^n$*,*
the period of p *jumps from* 2^m *to* 2^{m+1} *between* A_n *and* A_{n+1}
iff $j_{[p]}$ *maps* $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ *to* $\mathrm{crit}_\mathrm{n}(\mathrm{j})$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_\mathrm{m}(\mathrm{j}) = \mathrm{crit}(\mathrm{j}_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}(\mathrm{j}_{[p]}[\mathrm{j}_{[2^m]}])$,
so the RHT is $\mathrm{crit}(\mathrm{j}_{[p]}[\mathrm{j}_{[2^m]}]) = \mathrm{crit}_\mathrm{n}(\mathrm{j})$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.
On the other hand, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$, whence $\pi_n(p) \leqslant 2^m$.

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit_n}(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.
On the other hand, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$, whence $\pi_n(p) \leqslant 2^m$.
As $\pi_{n+1}(p)$ is $\pi_n(p)$ or $2\pi_n(p)$,

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- <u>Proposition</u> ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit_n(j)}$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m(j)} = \mathrm{crit(j_{[2^m]})}$, the embedding $j_{[p]}$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit(j_{[p]}[j_{[2^m]}])}$, so the RHT is $\mathrm{crit(j_{[p]}[j_{[2^m]}])} = \mathrm{crit_n(j)}$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$, which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.
On the other hand, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$, whence $\pi_n(p) \leqslant 2^m$.
As $\pi_{n+1}(p)$ is $\pi_n(p)$ or $2\pi_n(p)$, $(***)$ is equivalent to $\pi_n(p)=2^m$ and $\pi_{n+1}(p)=2^{m+1}$. □

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$,one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$

- <u>Lemma</u>: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

  - Proof: There exists β satisfying $j(\beta) > \alpha$,

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$,one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$
   - Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β,

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$

  - Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall\gamma < \beta\ (j(\gamma) \leqslant \alpha). \qquad (*)$$
Applying j to $(*)$ gives
$$\forall\gamma < j(\beta)\ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$,one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall\gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$
Applying j to $(*)$ gives
$$\forall\gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$
Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$,

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$,one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall \gamma < \beta \ (j(\gamma) \leqslant \alpha). \qquad (*)$$
Applying j to $(*)$ gives
$$\forall \gamma < j(\beta) \ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$
Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \ (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$. By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \ (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$. By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$. By the dictionary, j maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{\overline{n}}(j)$.

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*
$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$
Applying j to $(*)$ gives
$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$
Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\leqslant \mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\leqslant \mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Therefore, there exists $n' \leqslant \overline{n} \leqslant n$ s.t. $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{\mathrm{n}'}(\mathrm{j})$.

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\leqslant \mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Therefore, there exists $n' \leqslant \overline{n} \leqslant n$ s.t. $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{n'}(\mathrm{j})$.
By the dictionary, the period of 2 jumps from 2^m to 2^{m+1} between $A_{n'}$ and $A_{n'+1}$.

- <u>Lemma</u>: *If j belongs to E_λ, for every $\alpha < \lambda$,one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \ (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \ (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

- <u>Proposition</u> (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit_m(j)}$ to $\mathrm{crit_{\overline{n}}(j)}$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit_m(j)}$ to $\leqslant\mathrm{crit_{\overline{n}}(j)}$.
Therefore, there exists $n' \leqslant \overline{n} \leqslant n$ s.t. $j[j]$ maps $\mathrm{crit_m(j)}$ to $\mathrm{crit_{n'}(j)}$.
By the dictionary, the period of 2 jumps from 2^m to 2^{m+1} between $A_{n'}$ and $A_{n'+1}$.
Hence, the period of 2 in A_n is at least 2^{m+1}. □

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_n(j)$.

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_\mathrm{n}(\mathrm{j})$.
  - Not obvious: $\{\mathrm{crit}(\mathrm{i}) \mid \mathrm{i} \in \mathrm{Iter}(\mathrm{j})\}$ is countable, but its order type might be $>\omega$.

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_n(j)$.
  - Not obvious: $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_n(j)$.
  - Not obvious: $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit_n}(\mathrm{j})$.
  - Not obvious: $\{\mathrm{crit}(\mathrm{i}) \mid \mathrm{i} \in \mathrm{Iter}(\mathrm{j})\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$.

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_n(j)$.
  - Not obvious: $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_{\mathrm{n}}(\mathrm{j})$.
  - Not obvious: $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.

By the dictionary, this is equivalent to j mapping $\mathrm{crit}_{\mathrm{m}}(\mathrm{j})$ to $\mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals $\mathrm{crit}_n(j)$.*
  - Not obvious: $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.

By the dictionary, this is equivalent to j mapping $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{\overline{n}}(j)$.
Now j maps $\mathrm{crit}_m(j)$, which is $\mathrm{crit}(j_{[2^m]})$, to $\mathrm{crit}(j[j_{[2^m]}]$.
As $j[j_{[2^m]}]$ belongs to $\mathrm{Iter}(j)$, the lemma implies $\mathrm{crit}(j[j_{[2^m]}] = \mathrm{crit}_{\overline{n}}(j)$ for some $\overline{n}$. □

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit_n(j)}$.
  - Not obvious: $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.

  By the dictionary, this is equivalent to j mapping $\mathrm{crit_m(j)}$ to $\mathrm{crit_{\overline{n}}(j)}$.

  Now j maps $\mathrm{crit_m(j)}$, which is $\mathrm{crit(j_{[2^m]})}$, to $\mathrm{crit(j[j_{[2^m]}]}$.

  As $j[j_{[2^m]}]$ belongs to Iter(j), the lemma implies $\mathrm{crit(j[j_{[2^m]}]} = \mathrm{crit_{\overline{n}}(j)}$ for some $\overline{n}$. □

- Open questions: Find alternative proofs using no Laver cardinal.

- Are the properties of Laver tables an application of set theory?

- Are the properties of Laver tables an application of set theory?
  - So far, yes;

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition,

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements,

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting),
  one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements,
    then pass them to the mathematician for a formal proof.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal),

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables),

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal?

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal? (as done for the free shelf using the braid realization)

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal? (as done for the free shelf using the braid realization)
  - Can one use them in low-dimensional topology?

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal? (as done for the free shelf using the braid realization)
  - Can one use them in low-dimensional topology?

Richard Laver (1942-2012)

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathsf{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/\equiv_S$.

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathsf{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathsf{sp}}, \triangleright)/\equiv_S$. Does $\equiv_S$ extend to all of B_∞?

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/{\equiv_S}$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/{\equiv_S}$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Laver tables are quotients of the (free) set theoretic shelf $(\mathrm{Iter}(j), -[-])$.

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/\equiv_S$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Laver tables are quotients of the (free) set theoretic shelf $(\mathrm{Iter(j)}, -[-])$.

• Question 2: Can one find an alternative "self-iterating structure" $(S, \triangleright)$, which the Laver tables are quotients of?

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/\equiv_S$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Laver tables are quotients of the (free) set theoretic shelf $(\mathrm{Iter(j)}, -[-])$.

• Question 2: Can one find an alternative "self-iterating structure" $(S, \triangleright)$, which the Laver tables are quotients of?

- Typical candidate: Scott's domains in λ-calculus (?)

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\mathrm{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\mathrm{sp}}, \triangleright)/\equiv_S$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Laver tables are quotients of the (free) set theoretic shelf $(\mathrm{Iter(j)}, -[-])$.

• Question 2: Can one find an alternative "self-iterating structure" $(S, \triangleright)$, which the Laver tables are quotients of?

- Typical candidate: Scott's domains in λ-calculus (?)

• Question 3: Determine the (co)-homology of the free monogenrated shelf.

- Question ∞: Compute the function μ_n defined on B_n^+ (positive n-strand braids) by

$$\mu_n(\beta) := \inf\{\beta' \mid \beta' \text{ conjugated to } \beta\}.$$

$\uparrow$ standard linear braid ordering

- Question ∞: Compute the function μ_n defined on B_n^+ (positive n-strand braids) by

$$\mu_n(\beta) := \inf\{\beta' \mid \beta' \text{ conjugated to } \beta\}.$$

$\uparrow$ standard linear braid ordering

  - Remark: certainly doable, at least for $n = 3$.

- Question ∞: Compute the function μ_n defined on B_n^+ (positive n-strand braids) by

$$\mu_n(\beta) := \inf\{\beta' \mid \beta' \text{ conjugated to } \beta\}.$$

$\uparrow$ standard linear braid ordering

  - Remark: certainly doable, at least for $n = 3$.

- Question ∞': Same question with

$$\nu_n(\beta) := \inf\{\beta' \mid \beta' \text{ Markov-equivalent to } \beta\}.$$