Self-distributivity, braids, and set theory

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Self-distributive systems and quandle (co)homology theory
in algebra and low-dimensional topology, Pusan, June 2017

- Many things are known about shelves (SD-structures that need not be racks).
- Here special emphasis on the connection with braids and with set theory.

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
  - Quandles, racks, and shelves
- 2. The SD-world
  - Classical and exotic examples
  - The world of shelves
- 3. The braid shelf
  - The braid operation
  - Larue's lemma and free subshelves
  - Special braids
- 4. The free monogenerated shelf
  - Terms and trees
  - The comparison property
  - The Thompson's monoid of SD
- 5. The set-theoretic shelf
  - Set theory and large cardinals
  - Elementary embeddings
  - The iteration shelf
- 6. Using set theory to investigate Laver tables
  - Quotients of the iteration shelf
  - A dictionary
  - Results about periods

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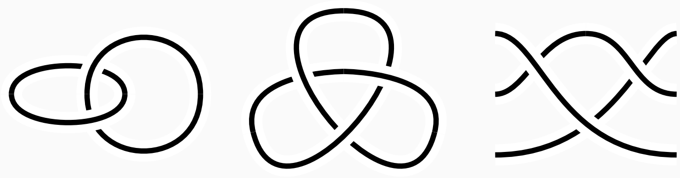
- Planar diagrams:



▶ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two diagrams are (projections of) isotopic figures

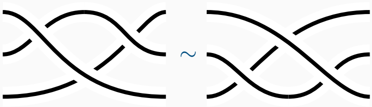
▶ find isotopy invariants.

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I :

- type II :

- type III :

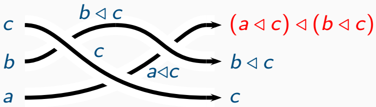


- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$, and color the strands in diagrams obeying the rules:

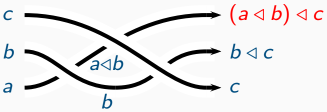
and

.

- Action of Reidemeister moves on colors:



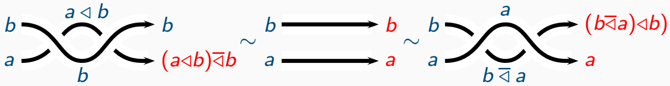
$\sim$



  - ▶ Hence: S-colorings invariant under Reidemeister move III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf

- Proposition: *Whenever $(S, \triangleleft)$ is a shelf, diagram coloring provides a well defined action of the braid monoid B_n^+ on S^n for every n.*

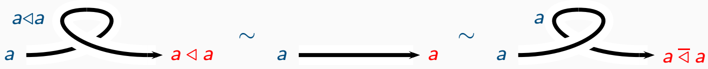
- Idem for Reidemeister move II:



- Lemma: *There exists* $\overline{\triangleleft}$ *satisfying* $(x \triangleleft y) \mathbin{\overline{\triangleleft}} y = x$ *and* $(x \mathbin{\overline{\triangleleft}} y) \triangleleft y = x$ *iff the right translations of* $(S, \triangleleft)$ *are bijections.*
  - Hence: S-colorings invariant under Reidemeister moves II+III $\Leftrightarrow$ $(S, \triangleleft)$ is a shelf with bijective right translations
    ↑ a rack

- Proposition: *Whenever* $(S, \triangleleft)$ *is a rack, diagram coloring provides a well defined action of the braid group* B_n *on* S^n *for every* n.

- Idem for Reidemeister move I:



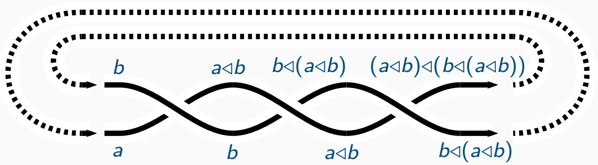
  - Hence: S-colorings invariant under Reidemeister moves I+II+III $\Leftrightarrow$ $(S, \triangleleft)$ is an idempotent rack
    ↑
    a quandle

- Theorem (Joyce, Matveev): *Define the fundamental quandle of the closure of an n-strand braid β to be*

$$\langle a_1, \ldots, a_n \mid a_1 = a'_1, \ldots, a_n = a'_n \rangle_{quandle}$$

where $a'_1, \ldots, a'_n$ are the output colors in (a diagram of) β with input colors $a_1, \ldots, a_n$. Then the fundamental quandle is a complete isotopy invariant up to mirror symmetry.

- Example:
The trefoil knot:



Leads to $\langle a, b \mid (b \triangleleft (a \triangleleft b) = a, (a \triangleleft b) \triangleleft (b \triangleleft (a \triangleleft b)) = b \rangle_{\text{quandle}}$,

i.e., $\langle a, b, c \mid a \triangleleft b = c, b \triangleleft c = a, c \triangleleft a = b \rangle_{\text{quandle}}$.

- Quandles and racks have being used successfully in knot theory
  in particular via homological approximations: Fenn, Rourke, Carter, Kamada ...

- Main question: Could shelves that are not racks be useful in topology?

- Bad news: General shelves are very different from racks.
  - Precise meaning: free racks are very special shelves...
  - Presumably much work to adapt the results. (?)
- Good news: General shelves are very different from racks.
  - If general shelves can be used, one can expect really new applications.

  - Explore the world of shelves...

- An example (of using non-rack shelves): the partial action of braids on a right-cancellative shelf

- Assume that $(S, \triangleleft)$ is a right-cancellative shelf

  $\uparrow$

  $a \triangleleft b = a' \triangleleft b \Rightarrow a = a'$: right translations are injective

Then one can define

and

the unique c s.t. $c \triangleleft a = b$, if it exists.

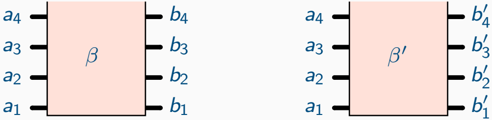
- <u>Proposition</u>: *One obtains in this way a well-defined partial action of B_n on S^n, s.t.*
  - *For all n-strand braid words $w_1, ..., w_p$, there exists at least one sequence $\vec{a}$ in S^n s.t. $\vec{a} \bullet w_i$ is defined for each i.*
  - *If w, w' are equivalent n-strand braid words and $\vec{a} \bullet w$ and $\vec{a} \bullet w'$ are defined, then $\vec{a} \bullet w = \vec{a} \bullet w'$ holds.*

  - Proof: Not trivial, uses the Garside structure of braids. □

$\rightsquigarrow$ a usable partial action...

- Definition: A shelf is orderable if there exists a linear ordering $<$ on S s.t.
  $a < b$ implies $a \triangleleft c < b \triangleleft c$, and $a < b \triangleleft a$ always holds.
  - Orderable shelves exist (see later...)
  - An orderable shelf is never a rack. If $(S, \triangleleft)$ is a rack:
    $$b \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft (a \triangleleft a) = ((b \,\overline{\triangleleft}\, a) \triangleleft a) \triangleleft a = b \triangleleft a,$$
    hence in particular $a \triangleleft a = a \triangleleft (a \triangleleft a)$. If $(S, \triangleleft)$ is orderable, $a \triangleleft a < a \triangleleft (a \triangleleft a)$.
  - An orderable shelf is right-cancellative: $a \neq b$ implies $a < b$ or $b < a$,
    whence $a \triangleleft c < b \triangleleft c$ or $a \triangleleft c > b \triangleleft c$, then $a \triangleleft c \neq b \triangleleft c$.

- Coloring braids using an orderable shelf directly provides a linear ordering of braids:



  - Then define $\beta < \beta'$ iff $\vec{a} \bullet \beta <^{\mathrm{Lex}} \vec{a} \bullet \beta'$.
    $\uparrow$
    $(b_1 < b'_1)$ or $(b_1 = b'_1$ and $b_2 < b'_2)$ or etc.

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- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$. (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t an element of R, E an R-module, and $x \triangleleft y := (1 - t)x + ty$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -2x + y$ ($\rightsquigarrow$ root systems).

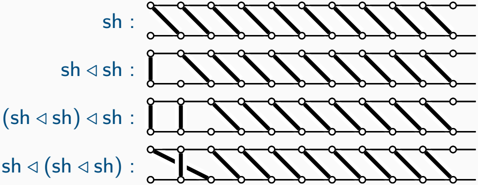
- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.

    $\uparrow$ when viewed as $(Q, \triangleleft, \overline{\triangleleft})$: $(F_X, \triangleleft)$ is not a free idempotent shelf, it satisfies other laws: $x \triangleleft (y \triangleleft (y \triangleleft x)) = (x \triangleleft (x \triangleleft y)) \triangleleft (y \triangleleft x)$, ... (Drápal-Kepka-Musílek, Larue)
  - Variants: $x \triangleleft y := y^{-n}xy^n$, $x \triangleleft y := f(y^{-1}x)y$ with $f \in \mathrm{Aut}(G)$, ...

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with $\mathrm{sh} : n \mapsto n+1$:



[P.D. *Algebraic properties of the shift mapping*, Proc. Amer. Math. Soc. 106 (1989) 617-623]

- Braid shelf: B_∞ braid group, $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$, and $\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta$.
  - Part 3 below
  - A variant: charged braids (realization of free shelves with $\geqslant 2$ generators)

    [P.D. *Construction of left distributive operations and charged braids*, Int. J. Alg. Comput 10 (2000) 173-190]
  - Another variant: transfinite braids (with a second, associative operation)

    [P.D. *Transfinite braids and left distributive operations*, Math. Z. 228 (1998) 405-433]
  - Another variant: parenthezised braids (aka Brin's braided Thompson's group $\widehat{BV}$)

    [P.D. *The group of parenthesized braids*, Adv. Math. 205 (2006) 354-409]

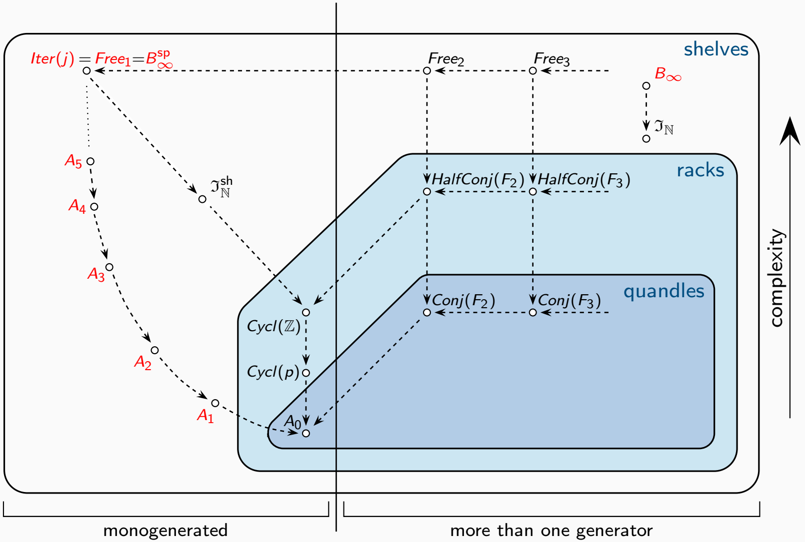
Transfinite braids and left distributive operations;

- Free shelves:
  - Case of one generator: Part 4 below
  - Case of $\geqslant 2$ generators: a lexicographic extension of the case of one generator

    [P.D. *A canonical ordering for free LD systems*, Proc. Amer. Math. Soc. 122 (1994) 31-37]

- Iteration shelf (set theory): λ a Laver cardinal, E_λ set of all elementary embeddings from V_λ to itself, and $i \triangleleft j := \bigcup_{\alpha<\lambda} j(i \cap V_\alpha^2)$
  - Part 5 below

- Laver tables: a family of finite shelves with 2^n elements
  - A. Drápal's minicourse
  - Part 6 below



shelves
racks
quandles
$Iter(j) = Free_1 = B_\infty^{sp}$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
A_5
A_4
A_3
A_2
A_1
A_0
$\mathfrak{I}_\mathbb{N}^{sh}$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator

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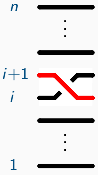
- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

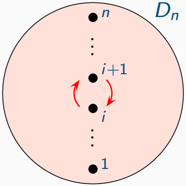
$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$



$\simeq$ mapping class group of D_n (disk with n punctures):

$\sigma_i \quad \leftrightarrow$

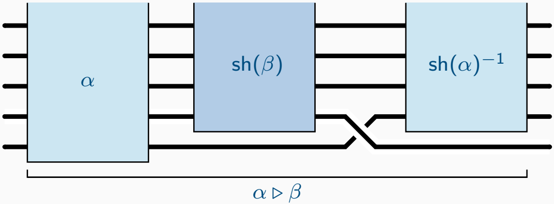


- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{matrix} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{matrix} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathrm{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathrm{sh}(\beta) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



- Examples: $1 \triangleright 1 = \sigma_1$, $1 \triangleright \sigma_1 = \sigma_2\sigma_1$, $\sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$, $\sigma_1 \triangleright \sigma_1 = \sigma_2\sigma_1$, etc.

▶ Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$.

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\,\mathrm{sh}(\gamma)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1})^{-1}$
$= \alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}\,\mathrm{sh}(\alpha)\,\mathrm{sh}^2(\gamma)\,\sigma_2\,\mathrm{sh}^2(\alpha)^{-1}\,\sigma_1\,\mathrm{sh}^2(\alpha)\,\sigma_2^{-1}\,\mathrm{sh}^2(\beta)^{-1}\,\mathrm{sh}(\alpha)^{-1}$
$= \alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}^2(\gamma)\,\sigma_2\sigma_1\sigma_2^{-1}\,\mathrm{sh}^2(\beta)^{-1}\,\mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$ □

• Remark: Shelf (=right shelf) with

$$\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta,$$

but less convenient here.

• Remark: Works similarly with

$$x \triangleright y := x \cdot \phi(y) \cdot e \cdot \phi(x)^{-1}$$

whenever G is a group G, e belongs to G, and ϕ is an endomorphism ϕ satisfying

$$e\,\phi(e)\,e = \phi(e)\,e\,\phi(e) \quad \text{and} \quad \forall x\,(e\,\phi^2(x) = \phi^2(x)\,e).$$

- Proposition (D., 1989, Laver, 1989) *If $(S, \triangleright)$ is a monogenerated left shelf, a sufficient condition for $(S, \triangleright)$ to be free is that the relation $\sqsubset$ on S has no cycle.*

  $\uparrow$ $x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

  - Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

- Theorem (D., 1991): *Every braid in B_∞ generates in $(B_\infty, \triangleright)$ a free left shelf.*

  - Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

  - Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$:
$\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Want to prove: $\rho(\alpha) \neq \rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n)$.
By definition: $\rho(\cdots((\alpha \triangleright \beta_1) \triangleright \beta_2) \triangleright \cdots) \triangleright \beta_n) = \rho(\alpha) \circ \rho(\gamma)$,
with γ a braid of the form $\mathrm{sh}(\gamma_0)\,\sigma_1\,\mathrm{sh}(\gamma_1)\,\sigma_1\,\mathrm{sh}(\gamma_2) \cdots \sigma_1\,\mathrm{sh}(\gamma_n)$, with no σ_1^{-1}.
Call such a braid **σ_1-positive**. It suffices to prove: "β is σ_1-positive $\Rightarrow \rho(\beta) \neq \mathrm{id}_{F_\infty}$".

- Lemma (Larue, 1992) *If β is σ_1-positive, then $\rho(\beta)(x_1)$ finishes with x_1^{-1}.*

▶ Proof: Identify F_∞ with the set of freely reduced words on $\{x_1, x_2, ...\}$ (no ss^{-1} or $s^{-1}s$). Use sh also for F_∞: $x_i \mapsto x_{i+1}$. Let

$$W := \{w \mid w \text{ reduced word finishing with } x_1^{-1}\}.$$

If β contains no $\sigma_1^{\pm 1}$, then $\rho(\beta)(x_1) = x_1$.
If $\beta = \sigma_1 \,\mathrm{sh}(\gamma)$, then $\rho(\beta)(x_1) = \rho(\sigma_1)(\rho(\mathrm{sh}(\gamma))(x_1)) = \rho(\sigma_1)(x_1) = x_1x_2x_1^{-1} \in W$.
So, it suffices to show: $w \in W$ implies $\rho(\sigma_1)(w) \in W$ and $\rho(\sigma_i^{\pm 1})(w) \in W$ for $i \geqslant 2$.

Assume $w \in W$, say $w = w'x_1^{-1}$, and consider $\rho(\sigma_1)(w) \in W$? Write ϕ for $\rho(\sigma_1)$.
Then $\phi(w) = \mathrm{red}(\phi(w')\, x_1 x_2^{-1} x_1^{-1})$. If $\phi(w)$ does not finish with x_1^{-1}, an x_1 in $\phi(w')$ cancels the final x_1^{-1}. This x_1 comes either from an x_1, or an x_1^{-1}, or an x_2 in w.

- For $w' = ux_1v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = 1$, then $v = 1$, and $w' = ux_1$, contradicting "$w'x_1^{-1}$ reduced".

- For $w' = ux_1^{-1}v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}x_1^{-1})$, with $\mathrm{red}(x_2^{-1}x_1^{-1}\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_1x_2^2x_1^{-1}$, then $v = x_1^2$, and $w' = ux_1^{-1}x_1^2$, contradicting "w' reduced".

- For $w' = ux_2v$, we find
  $\phi(w) = \mathrm{red}(\phi(u)x_1\phi(v)x_1x_2^{-1}x_1^{-1})$ with $\mathrm{red}(\phi(v)x_1x_2^{-1}) = 1$.
  Hence $\phi(v) = x_2^{-1}x_1$, then $v = x_2^{-1}x_1$, and $w' = ux_2x_2^{-1}x_1$, contradicting "w' reduced".□

- Definition: A braid β is special if it belongs to the closure of $\{1\}$ under $\triangleright$.
  - Examples: 1 is special; $1 \triangleright 1 = \sigma_1$ is special; $1 \triangleright (1 \triangleright 1) = 1 \triangleright \sigma_1 = \sigma_1\sigma_2$ is special; $(1 \triangleright 1) \triangleright 1 = \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$ is special, etc.

- Proposition: *Let B_∞^{sp} be the family of all special braids. Then $(B_\infty^{\mathrm{sp}}, \triangleright)$ is a realization of the free monogenerated left shelf.*

- Corollary ("word problem of SD"): *Two terms T, T' (in x and $\triangleright$) are SD-equivalent iff the braids $T(1)$ and $T'(1)$ evaluated in B_∞ are equal.*

- Lemma: *If β is a special braid, we have*

$$(1, 1, ...) \bullet \beta = (\beta, 1, 1, ...).$$

  - Proof: Induction on β special. True for 1. Then

$$\begin{aligned}(1, 1, ...) \bullet (\alpha \triangleright \beta) &= (((1, 1, ...) \bullet \alpha) \bullet \mathrm{sh}(\alpha)) \bullet \sigma_1) \bullet \mathrm{sh}(\beta)^{-1} \\ &= ((\alpha, 1, 1, ...) \bullet \mathrm{sh}(\beta)) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha, \beta, 1, ...) \bullet \sigma_1) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, \alpha, 1, ...) \bullet \mathrm{sh}(\alpha)^{-1} \\ &= (\alpha \triangleright \beta, 1, 1, ...)\end{aligned}$$

□

• Lemma: *For* $\vec{\alpha} = (\alpha_1, \alpha_2, ...)$ *in* $B_\infty^{(\mathbb{N})}$, *write* $\prod^{sh}\vec{\alpha}$ *for* $\alpha_1 \cdot sh(\alpha_2) \cdot sh^2(\alpha_3) \cdot \cdots$. *Then*

$$\vec{\alpha} \bullet \beta = \vec{\gamma} \quad \text{implies} \quad \prod^{sh}\vec{\alpha} \cdot \beta = \prod^{sh}\vec{\gamma}.$$

▶ Proof: Suffices to consider $\beta = \sigma_i^{\pm 1}$. Assume e.g. $\beta = \sigma_1$.
Then $\vec{\alpha}$ contributes $\alpha_1 \cdot sh(\alpha_2) \cdot \cdots$, whereas $\vec{\gamma}$ contributes
$\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot sh(\alpha_1)^{-1} \cdot sh(\alpha_1) \cdot \cdots$, i.e., $\alpha_1 \cdot sh(\alpha_2) \cdot \sigma_1 \cdot \cdots$.
As σ_1 commutes with every entry $sh^2(\alpha_i)$, that's OK. □

• Proposition: *Every braid* β *s.t.* $(1, 1, 1, ...) \bullet \beta$ *is defined admits a unique decomposition as* $\beta_1 \cdot sh(\beta_2) \cdot sh^2(\beta_3) \cdot \cdots$ *with* $\beta_1, \beta_2, ...$ *special.*

▶ Applies in particular to every positive braid.

▶ Proof: Assume $(1, 1, 1, ...) \bullet \beta = (\beta_1, \beta_2, \beta_3, ...)$. Then $\beta_1, \beta_2, ...$ are special.
As $\prod^{sh}(1, 1, 1...) = 1$, the lemma implies $\beta = \prod^{sh}\vec{\beta}$.
Conversely, assume $\beta = \prod^{sh}\vec{\beta}$. Then $(1, 1, 1, ...) \bullet \beta$ is defined, and it must be equal to $(\beta_1, \beta_2, ...)$, whence the uniqueness. □

.../...

[P.D. *Strange questions about braids*, J. Knot Th. Ramif. 8 (1999) 589-620]

• At this point, two main questions:

▶ Can one use the braid shelf and the associated diagram colorings in topology?
  ⇝ already used to define and investigate the braid ordering
  ⇝ new applications?

▶ Where does this (strange) operation come from?

Plan:

- 1. Braid colorings
  - Diagrams and Reidemeister moves
  - Diagram colorings
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- Describe the free (left) shelf based on a set X ($=$ the most general shelf gen'd by X)
  ($=$ the shelf generated by X, every shelf generated by X is a quotient of)

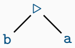
- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence (i.e., compatible equiv. rel.) on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

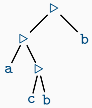
Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - Proof: trivial. □
  - ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X: assuming $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,



$\mathtt{b} \triangleright \mathtt{a}$



$(\mathtt{a} \triangleright (\mathtt{b} \triangleright \mathtt{c})) \triangleright \mathtt{b}$

- Lemma (confluence): *Let $\to_{SD}$ be the semi-congruence on $\mathcal{T}_X$ gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{SD} T_2$ holds iff one has $T_1 \to_{SD} T$ and $T_2 \to_{SD} T$ for some T.

"SD-equivalent iff admit a common SD-expansion"

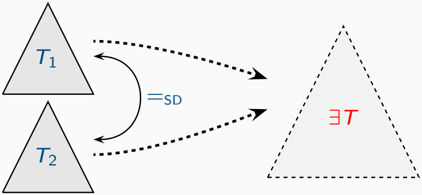
- Proof: $=_{SD}$ is the symmetric closure of $\to_{SD}$ (clear):

  $T_1 =_{SD} T_2$ holds iff there is a $\to_{SD}$-zigzag from T_1 to T_2.

  Suffices to show: if $T \to_{SD} T_1$ and $T \to_{SD} T_2$, then $\exists T' \, (T_1 \to_{SD} T'$ and $T_2 \to_{SD} T')$.

  Define $T \triangleright^* x := T \triangleright x$ and $T \triangleright^* (T_1 \triangleright T_2) := (T \triangleright^* T_1) \triangleright (T \triangleright^* T_2)$,
  and then $\partial x := x$ and $\partial(T_1 \triangleright T_2) := \partial T_1 \triangleright^* \partial T_2$. Then
  - $T \to_{SD} \partial T$ (easy induction),
  - $T \to_{SD}^{1} T'$ implies $T' \to_{SD} \partial T$ (semi-easy induction),
  - $T \to_{SD} T'$ implies $\partial T \to_{SD} \partial T'$ (more delicate induction).

  From there, $T \to_{SD}^{p} T'$ implies $T' \to_{SD} \partial^p T$ (easy), whence
  $T \to_{SD}^{p} T_1$ and $T \to_{SD}^{q} T_2$ implies $T_1 \to_{SD} \partial^r T$ and $T_2 \to_{SD} \partial^r T$ for $r \geqslant \max(p, q)$. □



- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n-1 > \mathrm{ht}(T_1)$ and $n-1 > \mathrm{ht}(T_2)$.

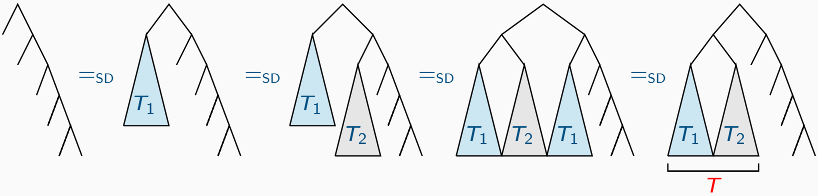
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ — by induction hypothesis for T_1

$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ — by induction hypothesis for T_2

$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ — by applying SD

$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]}$ — by induction hypothesis for T_1

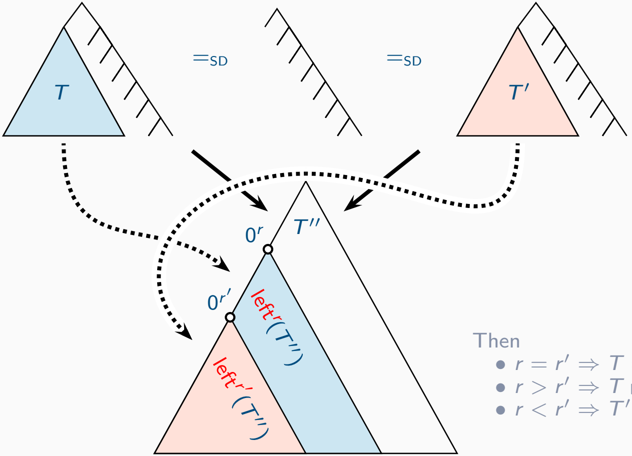
$= T \triangleright x^{[n]}$. □



- Lemma (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T'' \, (T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

▶ Proof:



Then
- $r = r' \Rightarrow T =_{SD} T'$
- $r > r' \Rightarrow T \sqsubset^*_{SD} T'$
- $r < r' \Rightarrow T' \sqsubset^*_{SD} T$

□

• <u>Lemma</u> (comparison II): *If* $(S, \triangleright)$ *is a monogenerated left-shelf, any two distinct elements of* S *are* $\sqsubset^*$*-comparable.*

$\uparrow$ transitive closure of $\sqsubset$ = iterated left divisibility relation

▶ Proof: Assume S gen'd by g and $a \neq a'$ in S. By def, $a = T(g)$ and $a' = T'(g)$ for some terms T, T'. If $T \sqsubset^*_{\mathrm{SD}} T'$, then $a \sqsubset^* a'$ in S; if $T' \sqsubset^*_{\mathrm{SD}} T$, then $a' \sqsubset^* a$ in S; otherwise, $T =_{\mathrm{SD}} T'$, hence $a = a'$ in S. □

• <u>Proposition</u> (freeness criterion): *If* $(S, \triangleright)$ *is a monogenerated left-shelf and* $\sqsubset$ *has no cycle, then* $(S, \triangleright)$ *is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$,

whence $T(g) \sqsubset^* T'(g)$ or $T'(g) \sqsubset^* T(g)$.

As $\sqsubset$ has no cycle in S, both imply $T(g) \neq T'(g)$. □

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

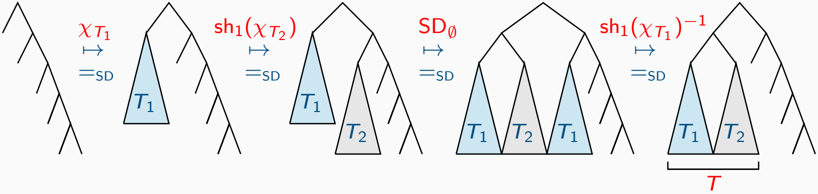
  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11\ldots1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11\ldots1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).
  - If ϕ maps T to T', then $\mathrm{sh}_0(\phi)$ maps $T \triangleright x^{[n]}$ to $T' \triangleright x^{[n]}$, so collapsing all $\mathrm{sh}_0(\phi)$ must give an SD-operation on the quotient, i.e., on B_∞.
  - Its definition is the projection of $(*)$, i.e.,

$$a \triangleright b := a \cdot \mathrm{sh}(b) \cdot \sigma_i \cdot \mathrm{sh}(a)^{-1}.$$

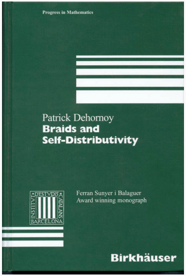
- The "magic rule" revisited:



$$\text{whence } \chi_{T_1 \triangleright T_2} = \chi_{T_1} \cdot sh_1(\chi_{T_2}) \cdot SD_\emptyset \cdot sh_1(\chi_{T_1}^{-1}),$$

which projects to the braid operation.

.../...



- See more in [P.D., Braids and selfdistributivity, PM192, Birkhaüser (1999)]

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- From the very beginning, Set Theory is a theory of infinity.

- Theorem (Cantor, 1873): *There exist* **at least two** *non-equivalent infinities.*

- Theorem (Cantor, 1880s): *There exist* **infinitely many** *non-equivalent infinities, which organize in a well-ordered sequence*

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\omega < \cdots .$$

- Facts: $\mathrm{card}(\mathbb{N}) = \aleph_0$, and $\mathrm{card}(\mathbb{R})$ $(= \mathrm{card}(\mathfrak{P}(\mathbb{N})) = 2^{\aleph_0})$ $> \mathrm{card}(\mathbb{N})$.

- Question: For which α (necessarily $\geqslant 1$) does $\mathrm{card}(\mathbb{R}) = \aleph_\alpha$ hold?
  - Conjecture (Continuum Hypothesis, Cantor, 1879): $\mathrm{card}(\mathbb{R}) = \aleph_1$.
  - Equivalently: every uncountable set of reals has the cardinality of $\mathbb{R}$.

- Beginning of XXth century: formalization of First Order logic (Frege, Russell, ...) and axiomatization of Set Theory (Zermelo, then Fraenkel, ZF):
  - Consensus: "*We agree that these properties express our current intuition of sets.*" (*but this may change in the future...*)
  - First question: Is CH or ¬CH (negation of CH) provable from ZF?

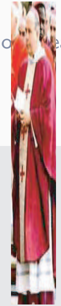
- Theorem (Gödel, 1938): *Unless* ZF *is contradictory,* ¬CH *cannot be proved from* ZF.

↑ negation of

- Theorem (Cohen, 1963): *Unless* ZF *is contradictory,* CH *cannot be proved from* ZF.

  - Method of proof: Investigate models of ZF = abstract structures satisfying the axioms of ZF (≈ investigate abstract groups or fields).
  - For Gödel: every model has a submodel that satisfies AC.
  - For Cohen: every model has an extension that satisfies ¬AC.

- Conclusion: ZF is incomplete (not: CH is indecidable—which means nothing).
  - Discover further properties of sets, and adopt an extended list of axioms!
  - How to recognize that an axiom is true? (What does this mean?)
    Example: CH may be taken as an additional axiom, but not a goo ea…



- Which new axioms?

- From 1930's, axioms of large cardinal:
  - various solutions to the equation

$$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$

  - set theory (as opposed to number theory) begins when "there exists an infinite set" is in the base axioms;
  - repeat the process with "super-infinite".
  - inaccessible cardinals, measurable cardinals, huge cardinals, ineffable cardinals, etc.

- Theorem(s) (Martin-Steel, Woodin, ... 1970s-80s): *A certain large cardinal axiom, PD ("projective determinacy", aka "there exists infinitely many Woodin cardinals"), provides a heuristically complete description of finite and countable sets.*

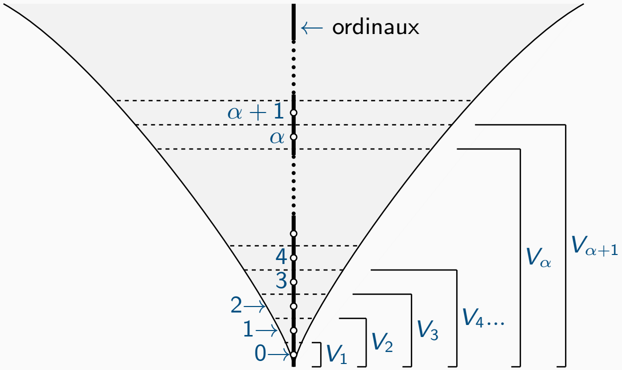
- New consensus: ZF+PD is, from now on, the reference system for set theory.

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - Proof: Assume $j : \mathbb{N} \to \mathbb{N}$ witnesses for "$\mathbb{N}$ is super-infinite".
    Then 0 is the only element of $\mathbb{N}$ satisfying "I am the smallest element for $<$",
    and $<$ is definable from $\in$. Hence $j(0)$ also satisfies "I am the smallest for $<$".
    Hence necessarily $j(0) = 0$. Now 1 says "I am the smallest after 0":
    By the same argument $j(1) = 1$, etc. So j is the identity. □
  - A super-infinite set must be so large that it contains undefinable elements
    (since all definable elements must be fixed).

- Fact: *There is a canonical filtration of sets by the sets V_α, α an ordinal, def'd by*
  $$V_0 := \emptyset, \quad V_{\alpha+1} := \mathfrak{P}(V_\alpha), \quad V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \text{ for } \lambda \text{ limit.}$$

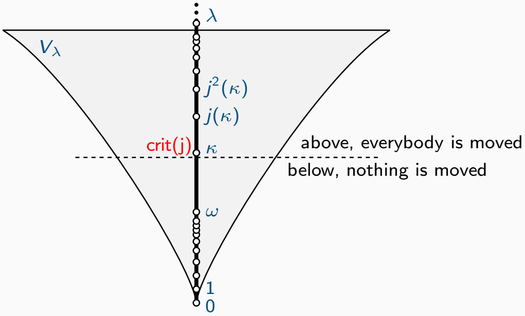


- Fact: *If λ is a limit ordinal and $f : V_\lambda \to V_\lambda$, then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ and $f \cap V_\alpha^2$ belongs to V_λ for every $\alpha < \lambda$.*
  - ▶ Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2, hence in $V_{\alpha+2}$, hence it belongs to $V_{\alpha+3}$, hence to V_λ. □

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit}(j))$.*



- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map $(i,j) \mapsto i[j]$ is a binary operation on E_λ, and $(E_\lambda, -[-])$ is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$ implies $\forall \alpha{<}j(\kappa)\, (j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$. We deduce $\mathrm{crit}(j[j]) = j(\kappa) > \kappa$. □

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf Iter(j).

↑

closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then Iter(j) is a free left-shelf.*

▶ Sketch of proof: Want to show that $i = i[i_1] \cdots [i_n]$ is impossible for $n \geqslant 1$. Consider here $n = 1$. Then $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) = \mathrm{i}(\mathrm{crit}(\mathrm{i}_1)) \in \mathrm{Im}(\mathrm{i})$, whereas $\mathrm{crit}(\mathrm{i}) \notin \mathrm{Im}(\mathrm{i})$. Hence $\mathrm{crit}(\mathrm{i}[\mathrm{i}_1]) \neq \mathrm{crit}(\mathrm{i})$, whence $i \neq i[i_1]$. □

- ▶ Another realization (the "set-theoretic realization") of the free (left)-shelf,
- ▶ ...plus a proof of that a left-shelf with acyclic $\sqsubset$ exists,
- ▶ ...whence a proof that $\sqsubset_{\mathrm{SD}}$ is acyclic on $\mathcal{T}_x$,
- ▶ ...whence a solution for the word problem of SD (because both $=_{\mathrm{SD}}$ and $\sqsubset^*_{\mathrm{SD}}$ are semi-decidable).

but all this under the (unprovable) assumption that a Laver cardinal exists.

⇝ motivation for finding another proof/another realization...

↑

the braid realization (1992)

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- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)}$:= the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  ▶ One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver): *Assume that j is an elementary embedding from V_λ to itself. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x \in V_\gamma\, (i(x) \cap V_\gamma = i'(x) \cap V_\gamma).$$

Then $\equiv_{\mathrm{crit_n(j)}}$ is a congruence on Iter(j), it has 2^n classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

▶ Proof: (Difficult...) Starts from $j \equiv_{\mathrm{crit(i)}} i[j]$ and similar.
Uses in particular $\mathrm{crit(j_{[m]})} = \mathrm{crit_n(j)}$ with n maximal s.t. 2^n divides m. □

- Recall: A_n is the unique left-shelf on $\{1, ..., 2^n\}$
  satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]}$ mod 2^n for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n*.*
  - ▶ Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
    The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
    By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.
    Then $j_{[2^n]} \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$ implies $j_{[2^n+1]} \equiv_{\mathrm{crit_n(j)}} j$, whence $2^n \triangleright 1 = 1$ in the quotient. □
  - ▶ A set-theoretic realization of A_n as a quotient of the (free) left-shelf Iter(j).

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- <u>Proposition</u> ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit}_n(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also $(p \triangleright 2^m)^{A_{n+1}} = 2^n$ $(***)$. First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.
On the other hand, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$, whence $\pi_n(p) \leqslant 2^m$.
As $\pi_{n+1}(p)$ is $\pi_n(p)$ or $2\pi_n(p)$, $(***)$ is equivalent to $\pi_n(p) = 2^m$ and $\pi_{n+1}(p) = 2^{m+1}$. □

• Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*

$$j(j)(\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and

$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$

Applying j to $(*)$ gives

$$\forall \gamma < j(\beta) \; (j(j)(\gamma) \leqslant j(\alpha)). \qquad (**)$$

Taking $\gamma := \alpha$ in $(**)$ yields $j(j)(\alpha) \leqslant j(\alpha)$. □

• Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{\overline{n}}(j)$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit}_m(j)$ to $\leqslant \mathrm{crit}_{\overline{n}}(j)$.
Therefore, there exists $n' \leqslant \overline{n} \leqslant n$ s.t. $j[j]$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{n'}(j)$.
By the dictionary, the period of 2 jumps from 2^m to 2^{m+1} between $A_{n'}$ and $A_{n'+1}$.
Hence, the period of 2 in A_n is at least 2^{m+1}. □

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit}_n(j)$.
  - Not obvious: $\{\mathrm{crit}(i) \mid i \in \mathrm{Iter}(j)\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.
  By the dictionary, this is equivalent to j mapping $\mathrm{crit}_m(j)$ to $\mathrm{crit}_{\overline{n}}(j)$.
  Now j maps $\mathrm{crit}_m(j)$, which is $\mathrm{crit}(j_{[2^m]})$, to $\mathrm{crit}(j[j_{[2^m]}]$.
  As $j[j_{[2^m]}]$ belongs to Iter(j), the lemma implies $\mathrm{crit}(j[j_{[2^m]}] = \mathrm{crit}_{\overline{n}}(j)$ for some $\overline{n}$. □

- Open questions: Find alternative proofs using no Laver cardinal.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal? (as done for the free shelf using the braid realization)
  - Can one use them in low-dimensional topology?

Richard Laver
(1942-2012)

• Question 0: Can shelves that are not racks be (really) useful in low-dimensional topology?

• Recall: $B_\infty^{\text{sp}} :=$ closure of $\{1\}$ under $\triangleright$ inside the infinite braid group B_∞ (realization of the free left shelf inside braids).

• Question 1: Let $(S, \triangleright)$ be a monogenerated (left) shelf. Find a concrete description of the congruence $\equiv_S$ s.t. $(S, \triangleright)$ is (isomorphic to) $(B_\infty^{\text{sp}}, \triangleright)/\equiv_S$. Does $\equiv_S$ extend to all of B_∞?

- Typical example: $S := A_n$, the nth Laver table.

• Laver tables are quotients of the (free) set theoretic shelf $(\text{Iter(j)}, -[-])$.

• Question 2: Can one find an alternative "self-iterating structure" $(S, \triangleright)$, which the Laver tables are quotients of?

- Typical candidate: Scott's domains in λ-calculus (?)

• Question 3: Determine the (co)-homology of the free monogenrated shelf.

- Question ∞: Compute the function μ_n defined on B_n^+ (positive n-strand braids) by

$$\mu_n(\beta) := \inf\{\beta' \mid \beta' \text{ conjugated to } \beta\}.$$

↑ standard linear braid ordering

  - Remark: certainly doable, at least for $n = 3$.

- Question ∞': Same question with

$$\nu_n(\beta) := \inf\{\beta' \mid \beta' \text{ Markov-equivalent to } \beta\}.$$