The SD-world: a bridge between algebra, topology, and set theory

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Fourth Mile High Conference, Denver, July-August 2017

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- 1. Overview of the SD-world, with a special emphasis on the word probleme of SD.

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- 1. Overview of the SD-world, with a special emphasis on the word probleme of SD.
- 2. The connection with set theory and the Laver tables.

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- Definition: A quandle is an idempotent rack ($x \triangleleft x = x$ always holds).

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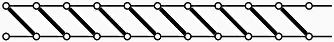
sh : 

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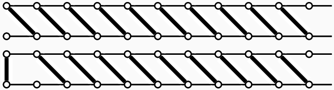

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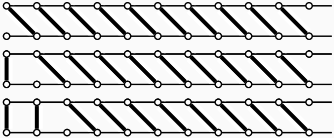

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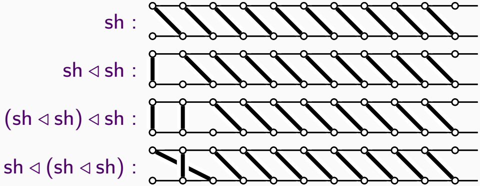

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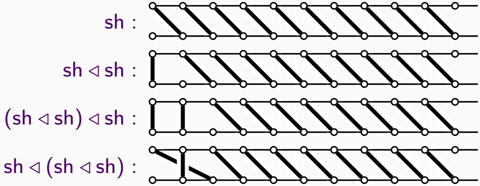

[P.D. *Algebraic properties of the shift mapping*, Proc. Amer. Math. Soc. 106 (1989) 617-623]

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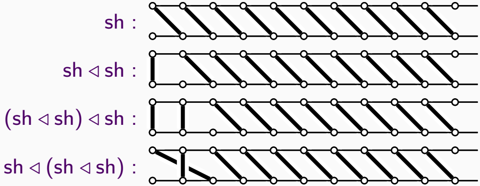

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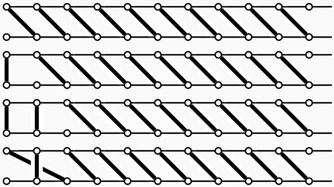

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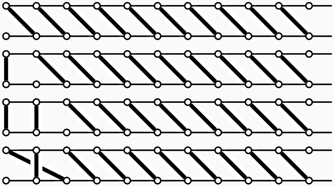

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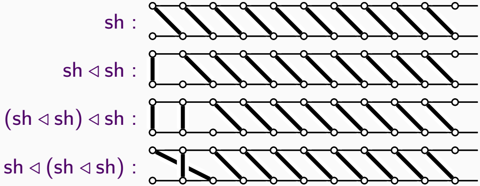
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with sh : $n \mapsto n + 1$:



[P.D. *Algebraic properties of the shift mapping*, Proc. Amer. Math. Soc. 106 (1989) 617-623]

- The braid shelf, the iteration shelf, Laver tables: see below...

- Planar diagrams:

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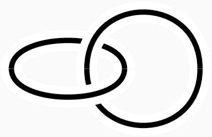
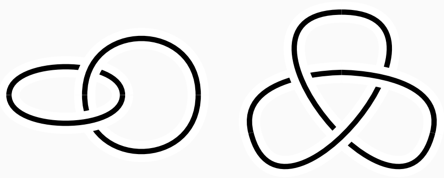
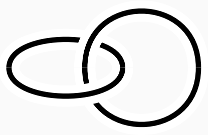
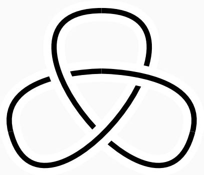


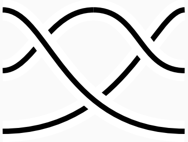
- Planar diagrams:



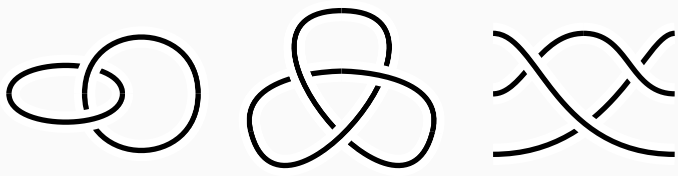
- Planar diagrams:





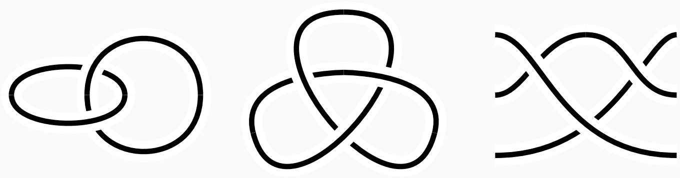


- Planar diagrams:



  - ▸ projections of curves embedded in $\mathbb{R}^3$

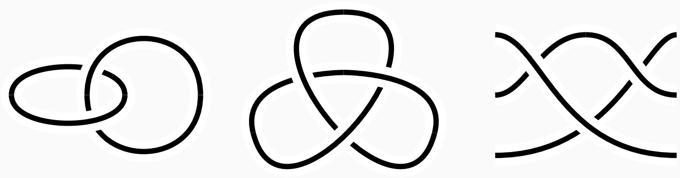
- Planar diagrams:



▸ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two 2D-diagrams are (projections of) isotopic 3D-figures

- Planar diagrams:

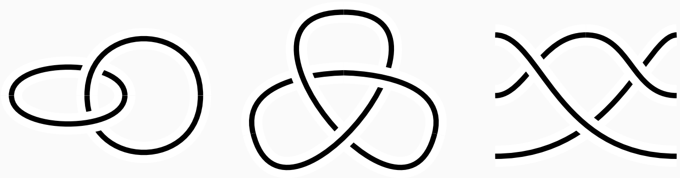


▸ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two 2D-diagrams are (projections of) isotopic 3D-figures

↑ continuously deform the 3D-figure allowing no curve crossing

- Planar diagrams:



▶ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two 2D-diagrams are (projections of) isotopic 3D-figures

↑ continuously deform the 3D-figure allowing no curve crossing

▶ find isotopy invariants.

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

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- type I :

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- type I :

- type II :

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- type I :

- type II :

Connection with topology (2)

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- type I :

- type II :

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- type I :

- type II :

- type III :

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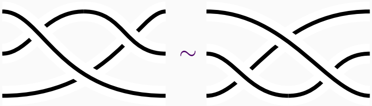
- type I :

- type II :

- type III :



- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,

- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$,
  and color the strands in diagrams obeying the rules:

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and

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- Action of Reidemeister moves on colors:

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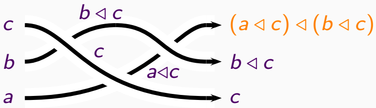
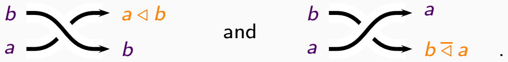


- Fix a set (of colors) S equipped with two operations $\triangleleft, \overline{\triangleleft}$, and color the strands in diagrams obeying the rules:



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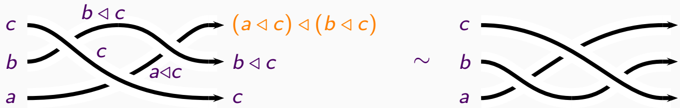

- Fix a set (of colors) S equipped with two operations $\lhd, \overline{\lhd}$,
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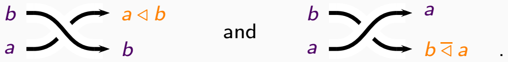

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and

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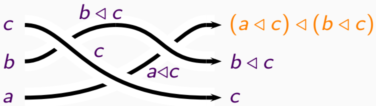

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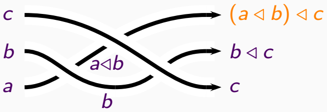
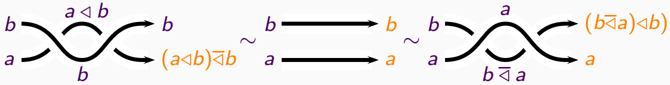


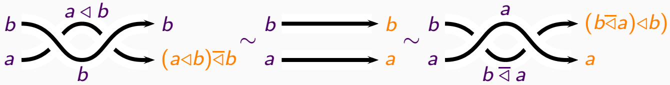
  - Hence:

$(S, \triangleleft)$-colorings are invariant under Reidemeister move III iff $(S, \triangleleft)$ is a shelf.

- Idem for Reidemeister move II:



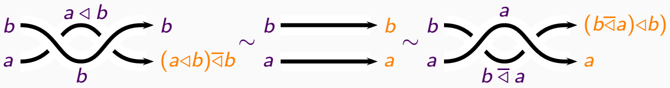
- Idem for Reidemeister move II:



  - Hence:

$(S, \triangleleft)$-colorings are invariant under Reidemeister moves II+III iff $(S, \triangleleft)$ is a rack.

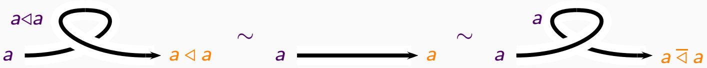
- Idem for Reidemeister move II:



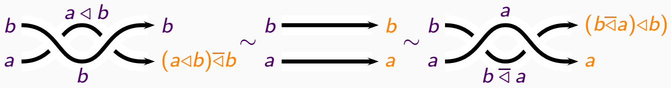
  - Hence:

$(S, \triangleleft)$-colorings are invariant under Reidemeister moves II+III iff $(S, \triangleleft)$ is a rack.

- Idem for Reidemeister move I:



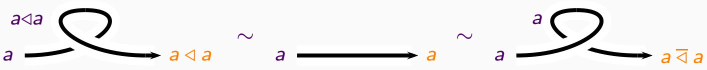
- Idem for Reidemeister move II:



  - Hence:

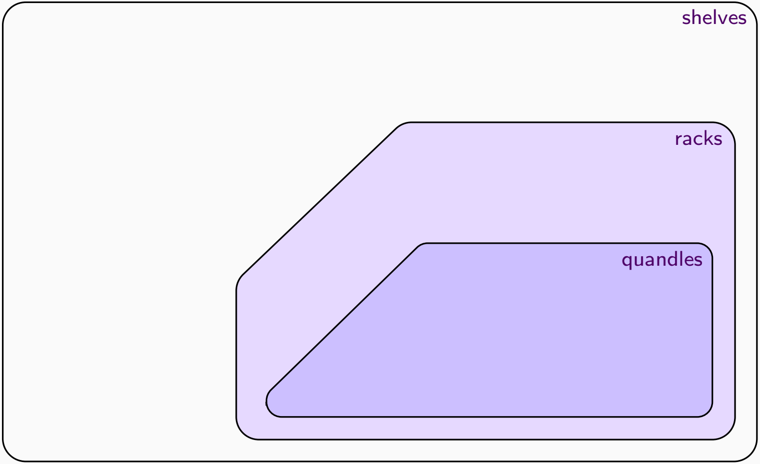
$(S, \triangleleft)$-colorings are invariant under Reidemeister moves II+III iff $(S, \triangleleft)$ is a rack.

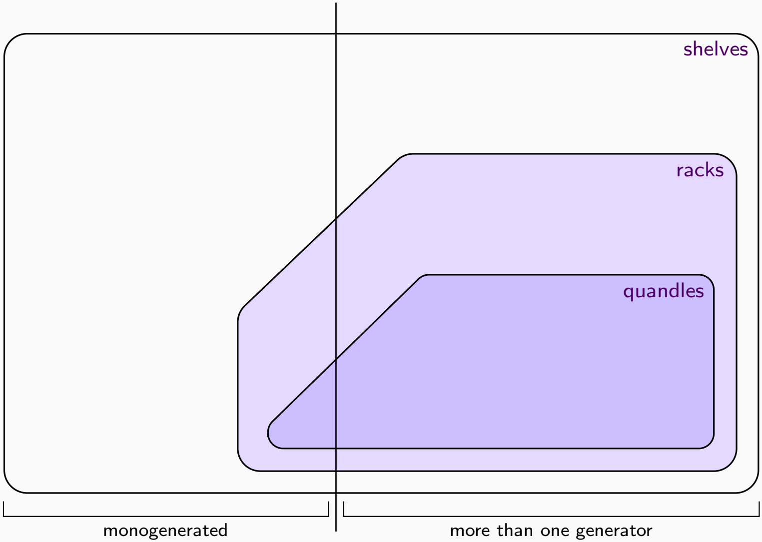
- Idem for Reidemeister move I:

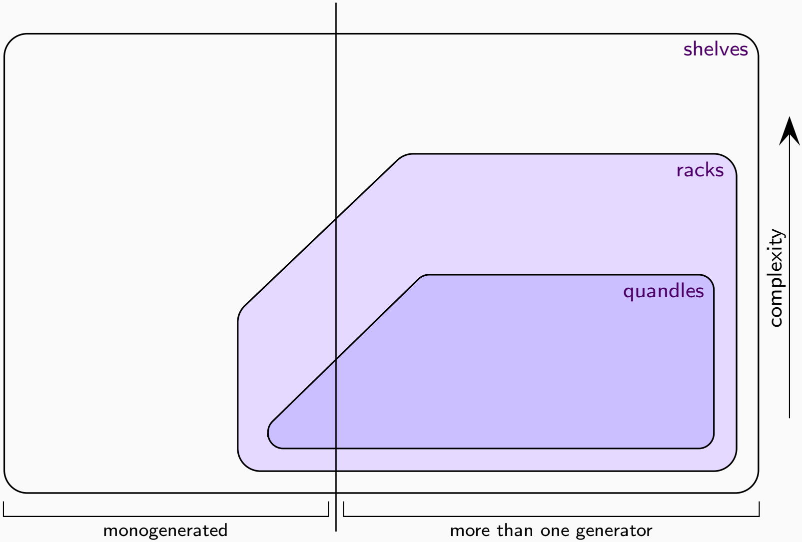


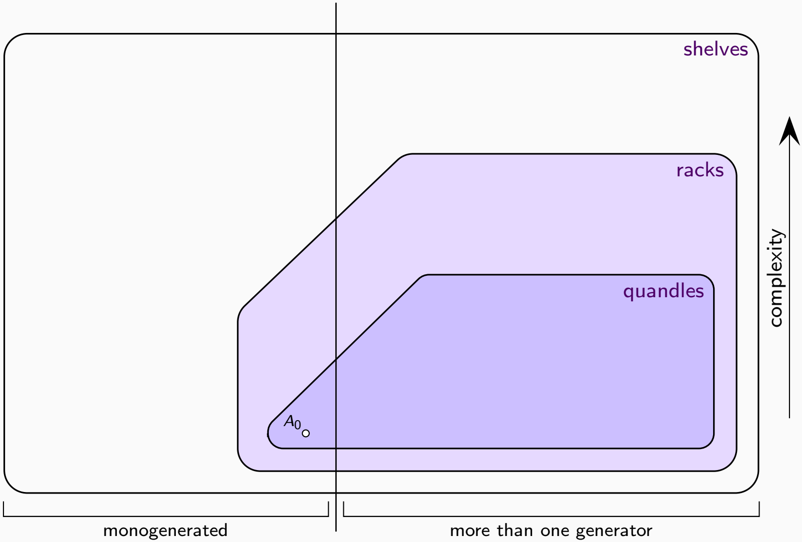
  - Hence:

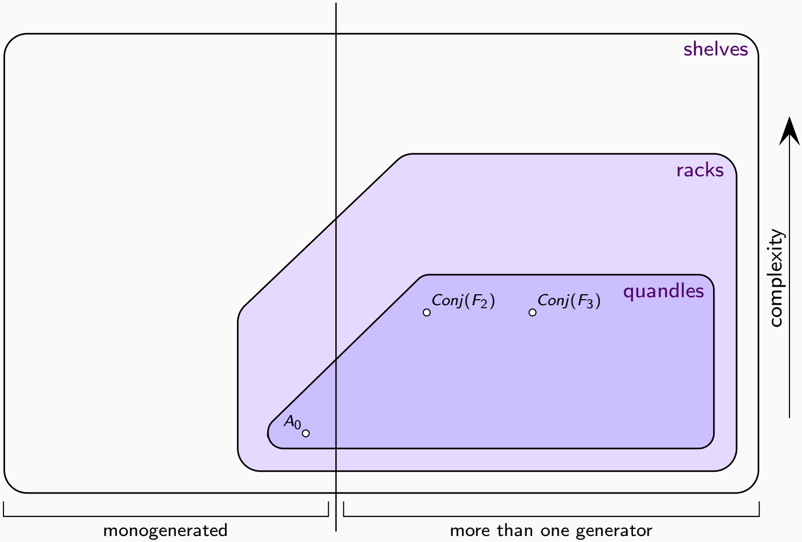
$(S, \triangleleft)$-colorings are invariant under Reidemeister moves I+II+III iff $(S, \triangleleft)$ is a quandle.

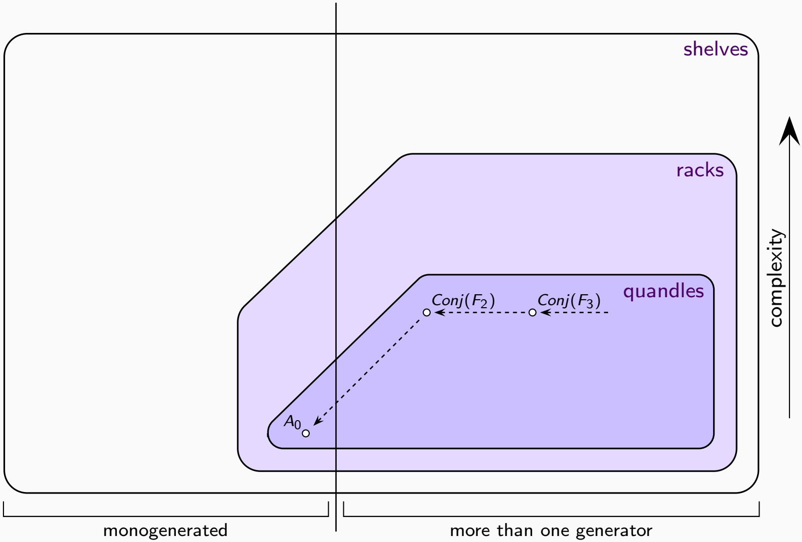

shelves
racks
quandles


shelves
racks
quandles
monogenerated
more than one generator


shelves
racks
quandles
complexity
monogenerated
more than one generator


shelves
racks
quandles
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Conj(F_2)$
$Conj(F_3)$
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
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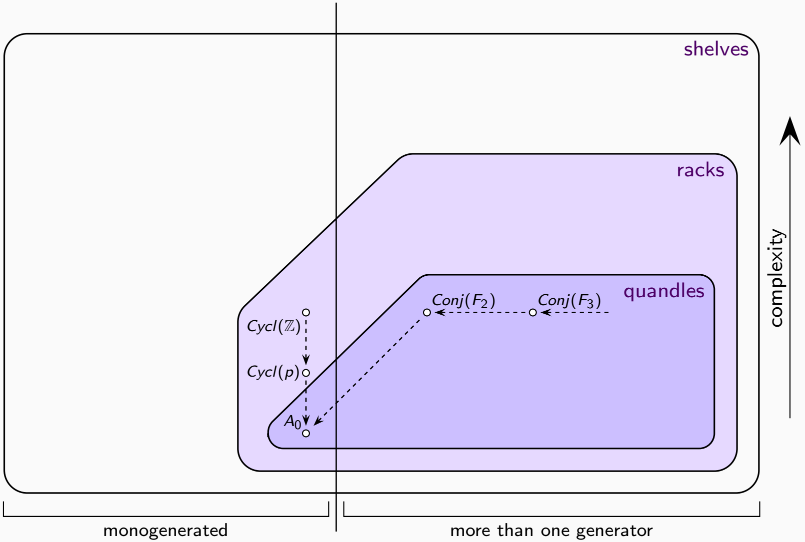
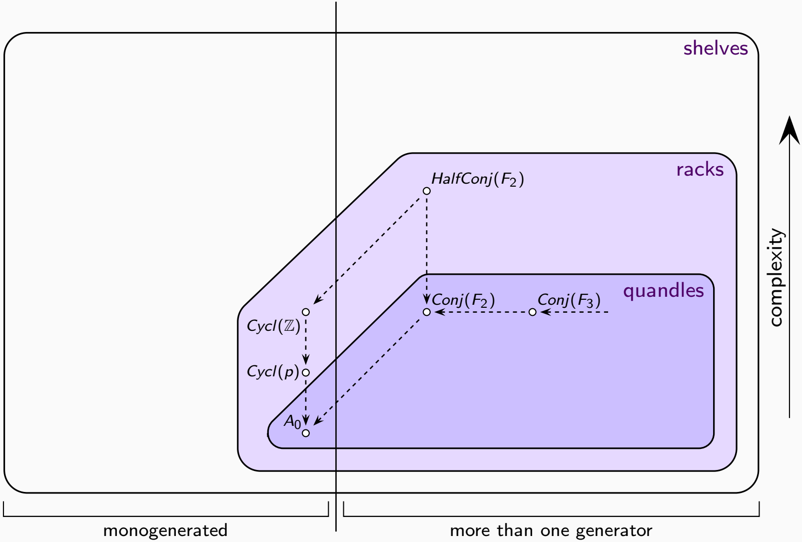
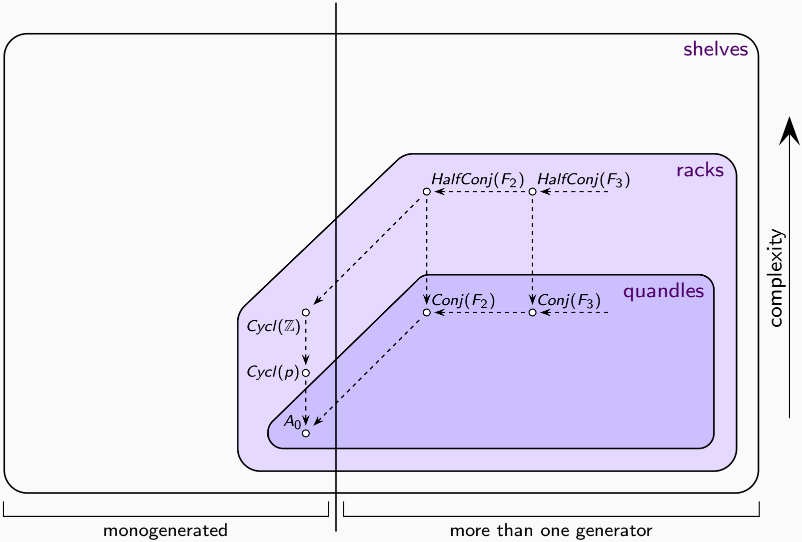
$Conj(F_3)$
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
HalfConj(F2)
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
A0
complexity
monogenerated
more than one generator


shelves
racks
quandles
HalfConj(F_2)
HalfConj(F_3)
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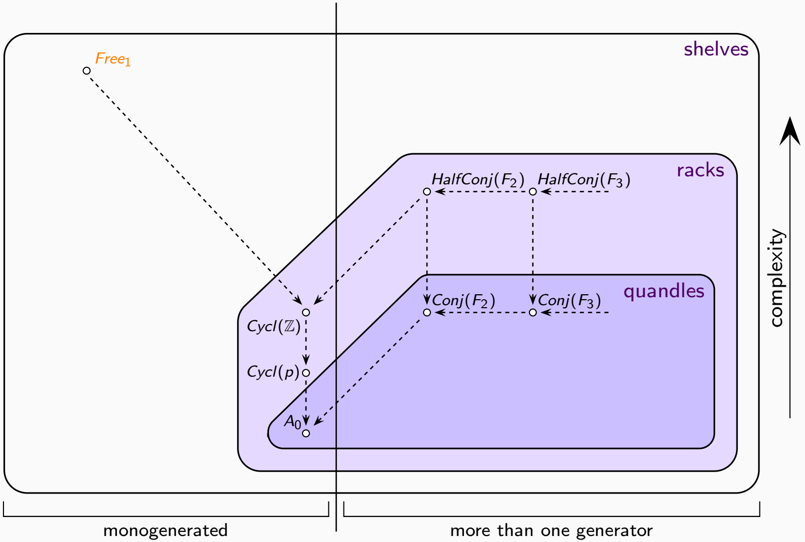
Conj(F_3)
Cycl($\mathbb{Z}$)
Cycl(p)
A_0
complexity
monogenerated
more than one generator


shelves
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racks
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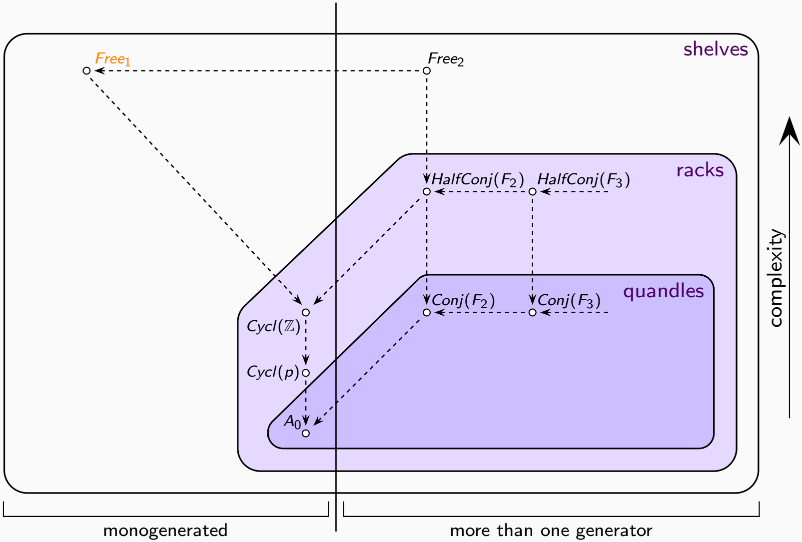
$HalfConj(F_3)$
quandles
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
complexity
monogenerated
more than one generator



shelves
$Free_1$
$Free_2$
racks
$HalfConj(F_2)$
$HalfConj(F_3)$
quandles
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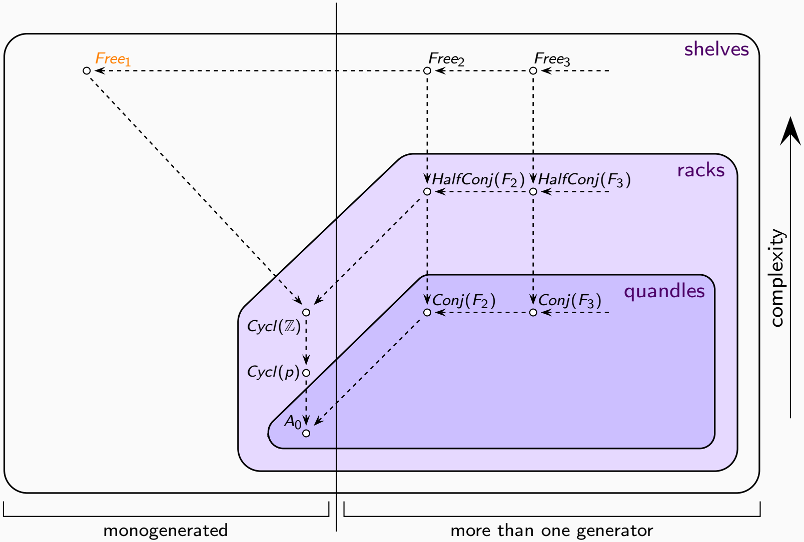
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Free_1$
$Free_2$
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$HalfConj(F_2)$
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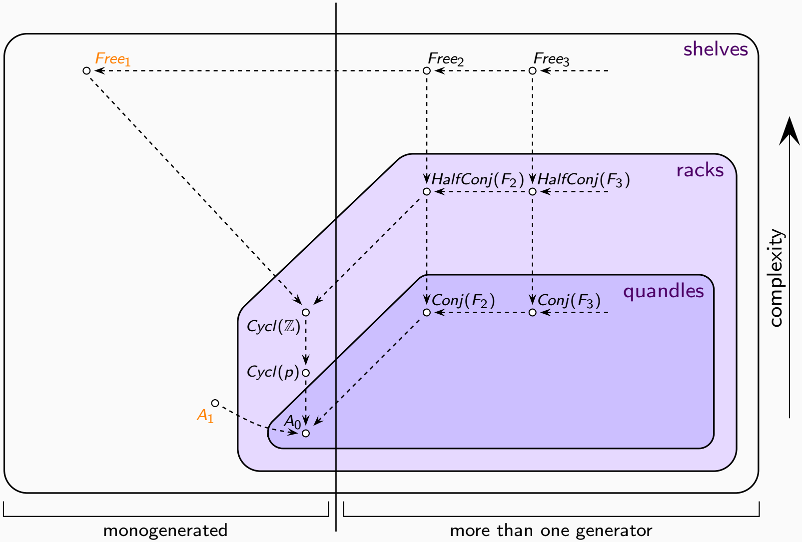
$Cycl(p)$
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Free_1$
$Free_2$
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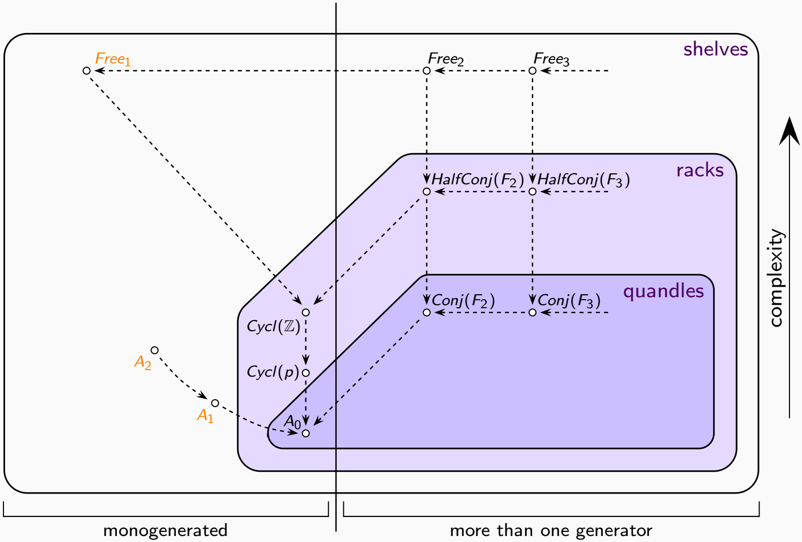
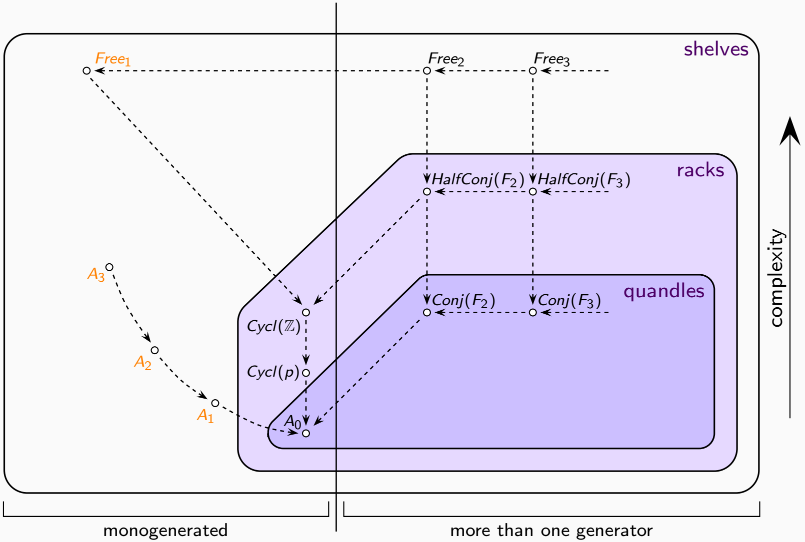
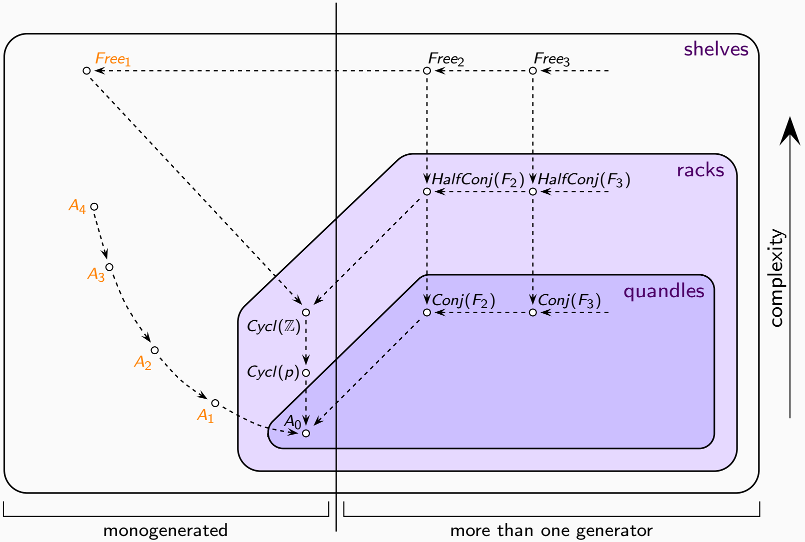
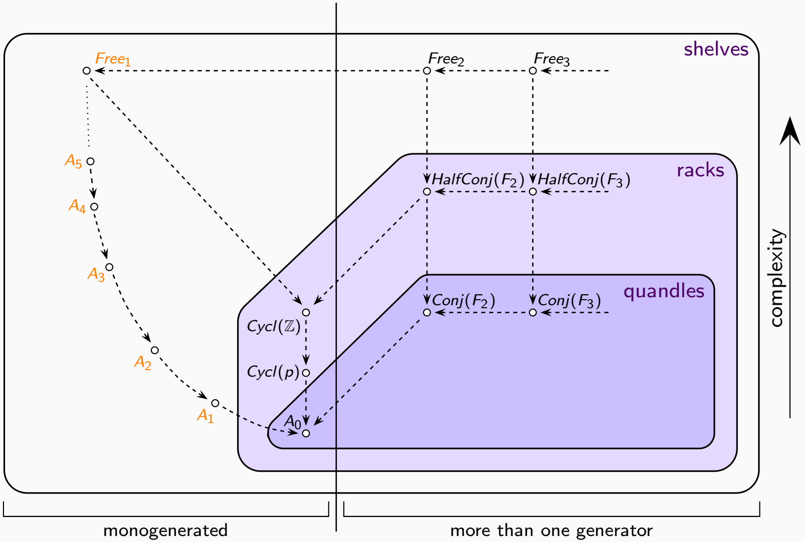
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_1
A_0
complexity
monogenerated
more than one generator

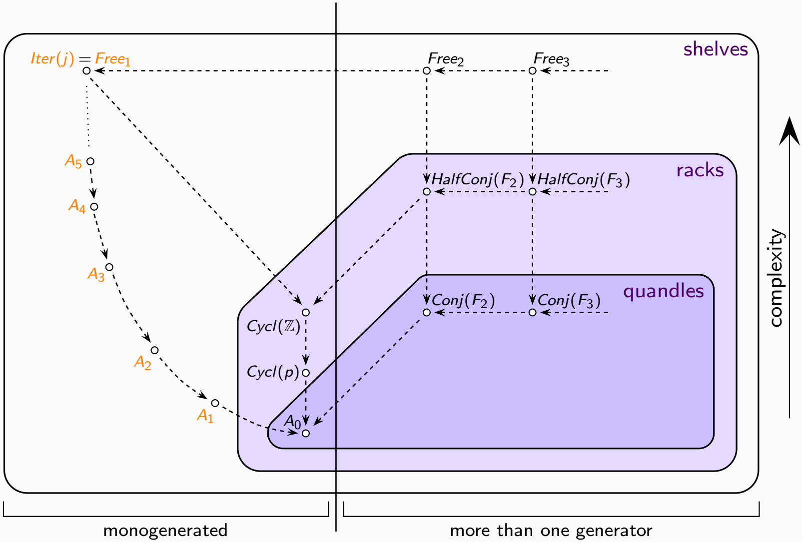


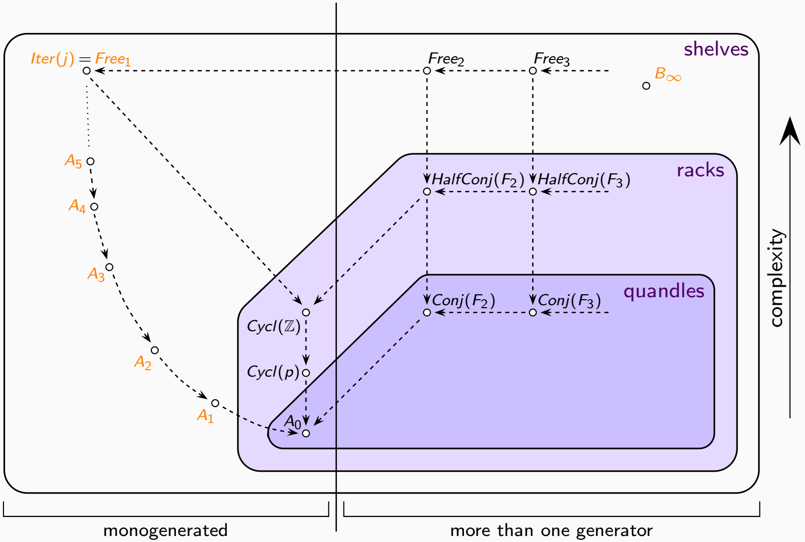
shelves
racks
quandles
$Free_1$
$Free_2$
$Free_3$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_2
A_1
A_0
complexity
monogenerated
more than one generator

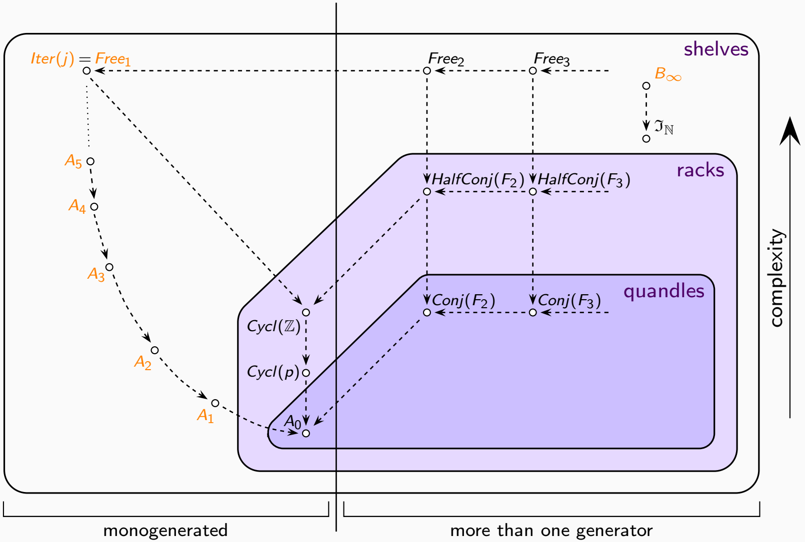

shelves
$Free_1$
$Free_2$
$Free_3$
racks
$HalfConj(F_2)$
$HalfConj(F_3)$
quandles
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_3
A_2
A_1
A_0
complexity
monogenerated
more than one generator


shelves
racks
quandles
$Free_1$
$Free_2$
$Free_3$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
A_0
A_1
A_2
A_3
A_4
complexity
monogenerated
more than one generator

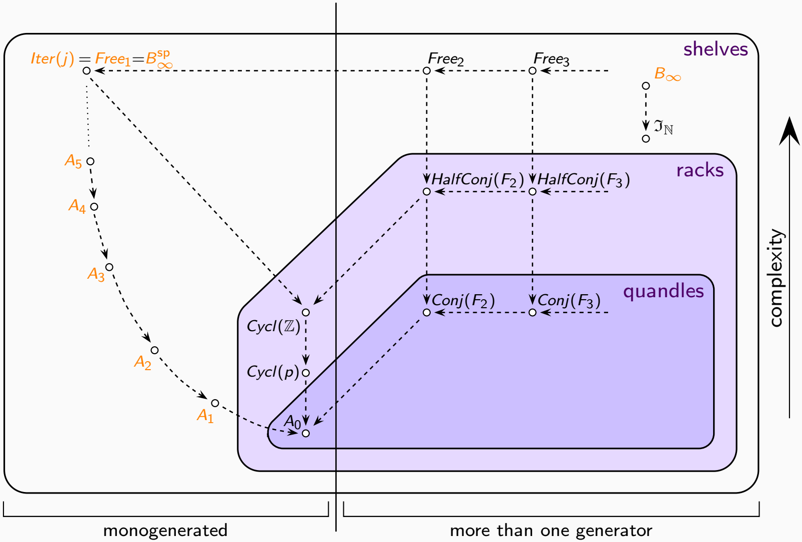

shelves
racks
quandles
complexity
Free$_1$
Free$_2$
Free$_3$
A_5
A_4
A_3
A_2
A_1
A_0
HalfConj(F_2)
HalfConj(F_3)
Conj(F_2)
Conj(F_3)
Cycl($\mathbb{Z}$)
Cycl(p)
monogenerated
more than one generator


shelves
racks
quandles
$Iter(j) = Free_1$
$Free_2$
$Free_3$
A_5
A_4
A_3
A_2
A_1
A_0
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator

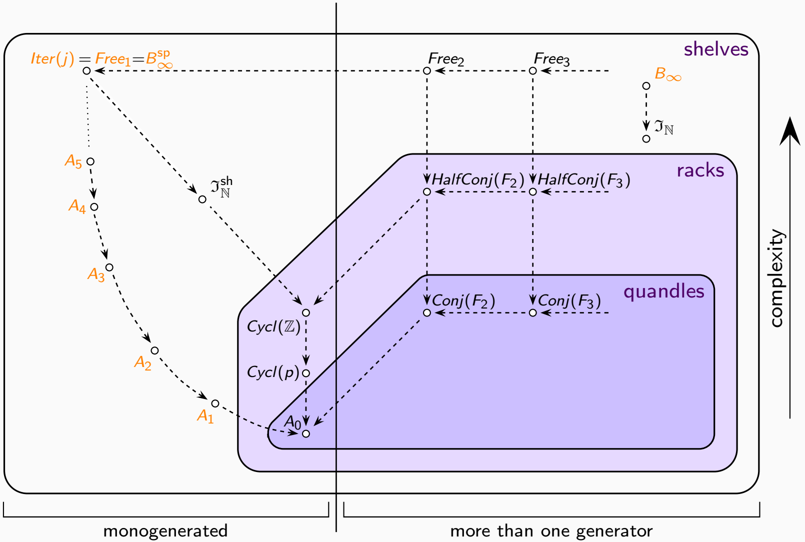

shelves
racks
quandles
Iter(j) = Free1
Free2
Free3
B∞
A5
A4
A3
A2
A1
A0
HalfConj(F2)
HalfConj(F3)
Conj(F2)
Conj(F3)
Cycl(ℤ)
Cycl(p)
complexity
monogenerated
more than one generator



shelves
racks
quandles
$Iter(j) = Free_1$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
A_5
A_4
A_3
A_2
A_1
A_0
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator



shelves
racks
quandles
$Iter(j) = Free_1 = B_\infty^{sp}$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
A_5
A_4
A_3
A_2
A_1
A_0
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator



shelves
racks
quandles
$Iter(j) = Free_1 = B_\infty^{sp}$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
$\mathfrak{I}_\mathbb{N}^{sh}$
A_5
A_4
A_3
A_2
A_1
A_0
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator

Plan:

- Minicourse I. The SD-world
  - 1. A general introduction
    - Classical and exotic examples
    - Connection with topology: quandles, racks, and shelves
    - A chart of the SD-world
  - 2. The word problem of SD: a semantic solution
    - Braid groups
    - The braid shelf
    - A freeness criterion
  - 3. The word problem of SD: a syntactic solution
    - The free monogenerated shelf
    - The comparison property
    - The Thompson's monoid of SD
- Minicourse II. Connection with set theory
  - 1. The set-theoretic shelf
    - Large cardinals and elementary embeddings
    - The iteration shelf
  - 2. Periods in Laver tables
    - Quotients of the iteration shelf
    - The dictionary
    - Results about periods

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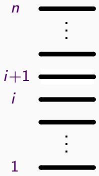

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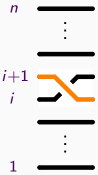

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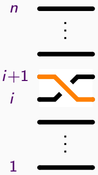

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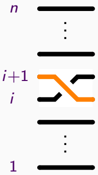

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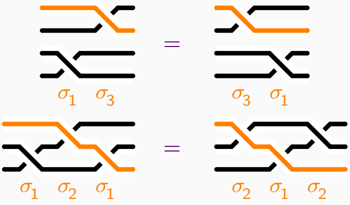

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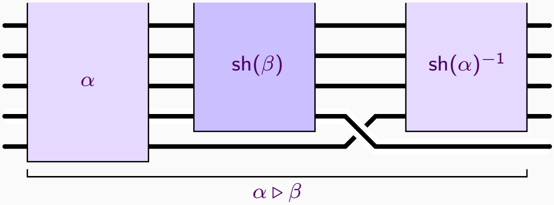

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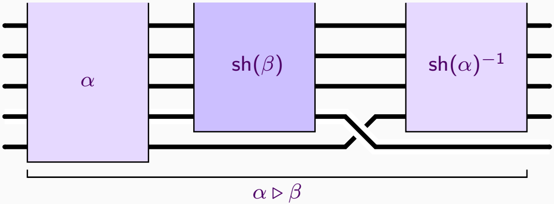

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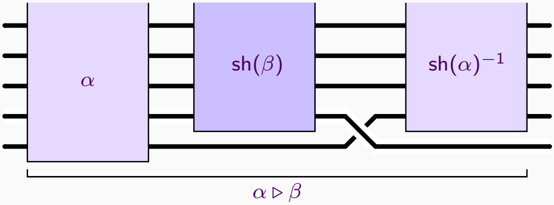

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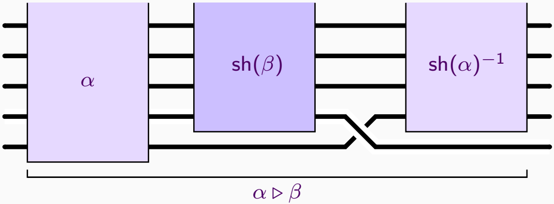

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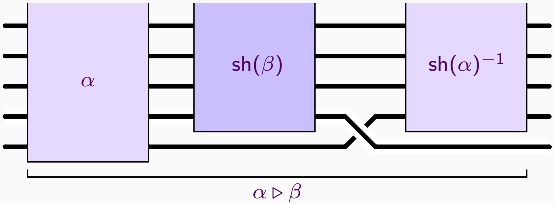

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• Remark: Shelf (=right shelf) with

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$$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$$

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$$= \alpha\,\mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\alpha)^{-1}\,\mathrm{sh}(\alpha)\,\mathrm{sh}^2(\gamma)\,\sigma_2\,\mathrm{sh}^2(\alpha)^{-1}\,\sigma_1\,\mathrm{sh}^2(\alpha)\,\sigma_2^{-1}\,\mathrm{sh}^2(\beta)^{-1}\,\mathrm{sh}(\alpha)^{-1}$$

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$$x \triangleright y := x \cdot \phi(y) \cdot e \cdot \phi(x)^{-1}$$

whenever G is a group G, e belongs to G, and ϕ is an endomorphism ϕ satisfying

$$e\,\phi(e)\,e = \phi(e)\,e\,\phi(e) \quad \text{and} \quad \forall x\,(e\,\phi^2(x) = \phi^2(x)\,e).$$

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Plan:

- Minicourse I. The SD-world
  - 1. A general introduction
    - Classical and exotic examples
    - Connection with topology: quandles, racks, and shelves
    - A chart of the SD-world
  - 2. The word problem of SD: a semantic solution
    - Braid groups
    - The braid shelf
    - A freeness criterion
  - 3. The word problem of SD: a syntactic solution
    - The free monogenerated shelf
    - The comparison property
    - The Thompson's monoid of SD
- Minicourse II. Connection with set theory
  - 1. The set-theoretic shelf
    - Large cardinals and elementary embeddings
    - The iteration shelf
  - 2. Periods in Laver tables
    - Quotients of the iteration shelf
    - The dictionary
    - Results about periods

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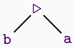

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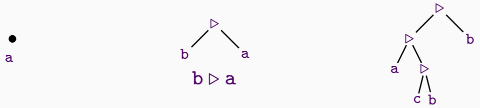

$\mathtt{b} \triangleright \mathtt{a}$ $\qquad$ $(\mathtt{a} \triangleright (\mathtt{b} \triangleright \mathtt{c})) \triangleright \mathtt{b}$

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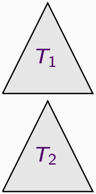

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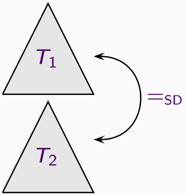

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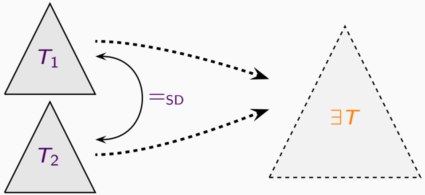

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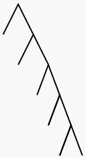

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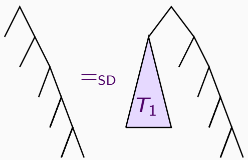

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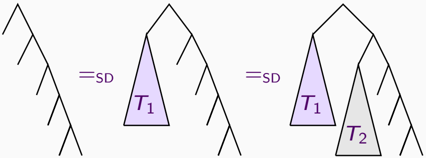

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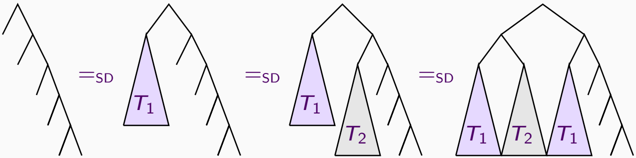

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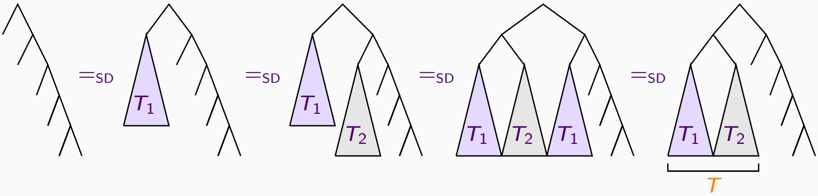

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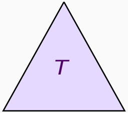

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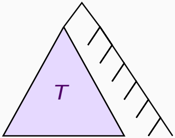

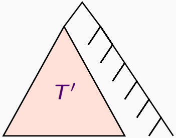

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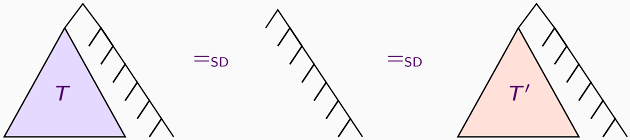

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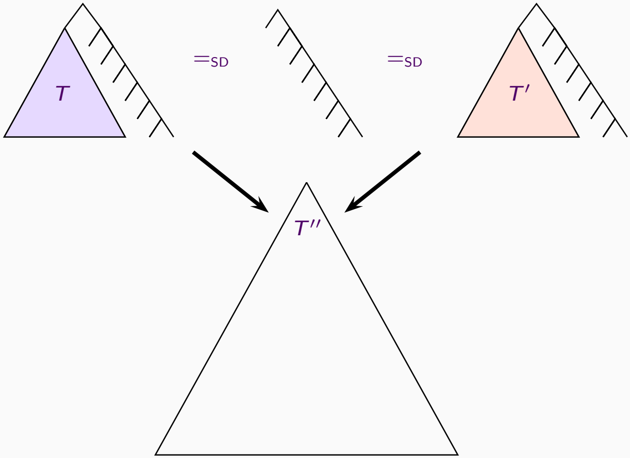

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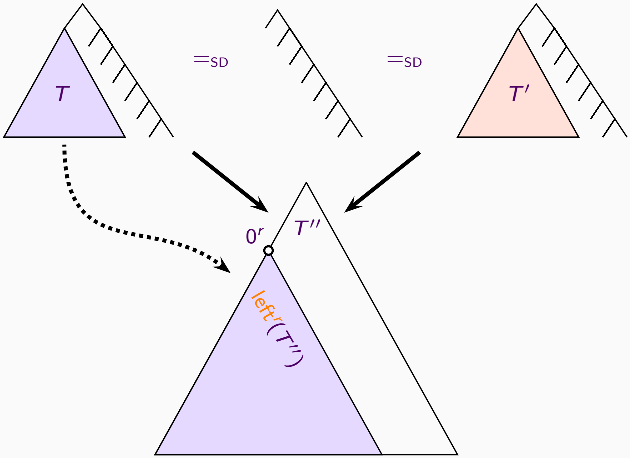

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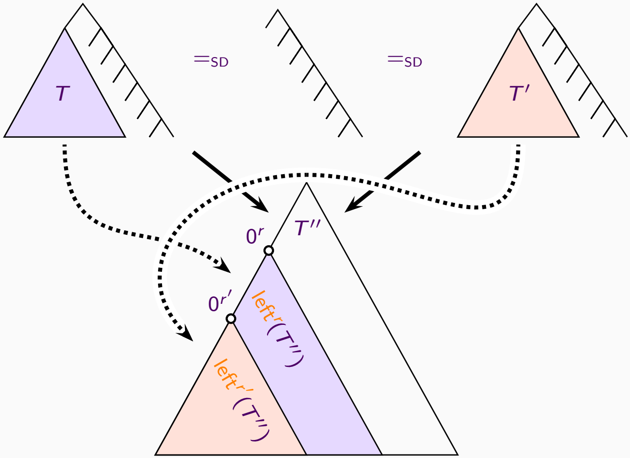

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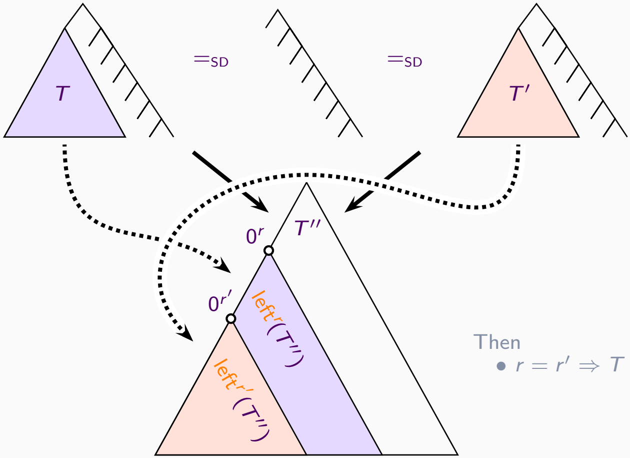

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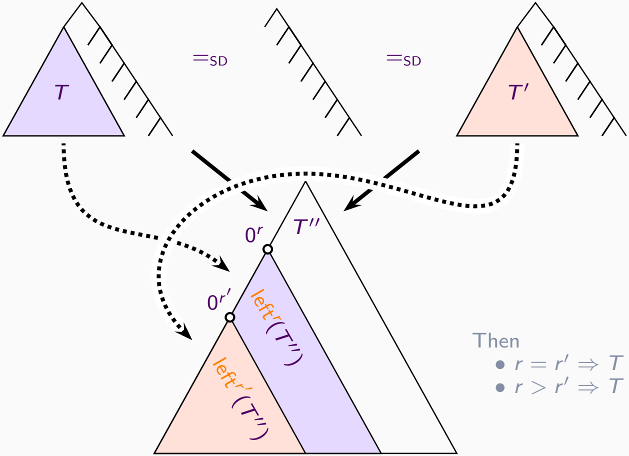

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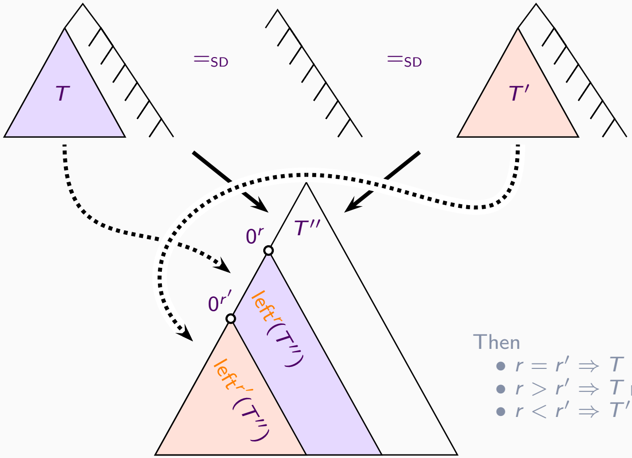

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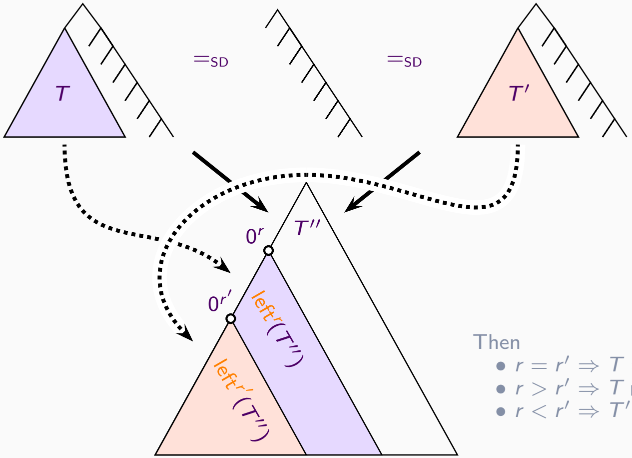

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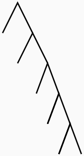

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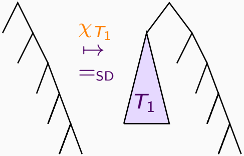

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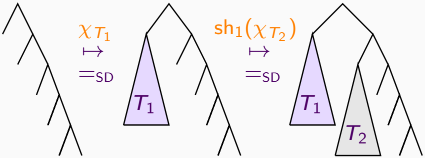

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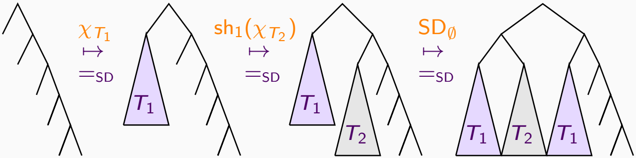

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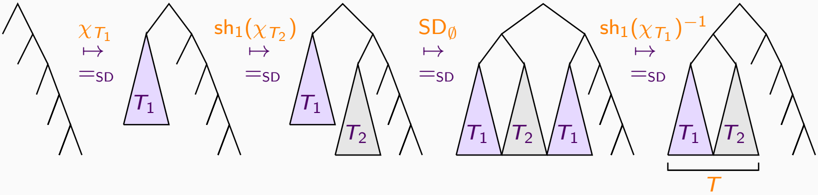

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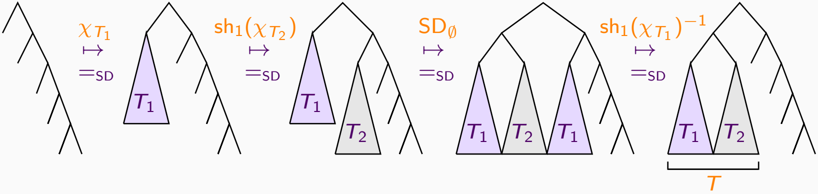

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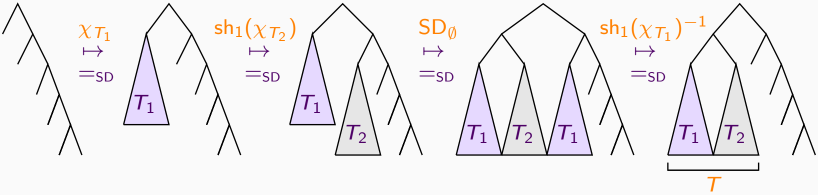

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The Thompson's monoid of SD (cont'd)

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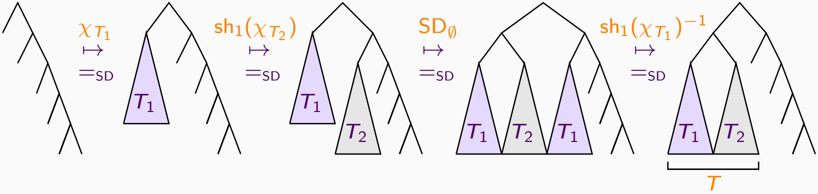

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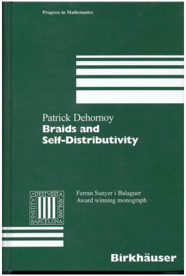
- See more in [P.D., Braids and selfdistributivity, PM192, Birkhaüser (1999)]



Plan:

- Minicourse I. The SD-world
  - 1. A general introduction
    - Classical and exotic examples
    - Connection with topology: quandles, racks, and shelves
    - A chart of the SD-world
  - 2. The word problem of SD: a semantic solution
    - Braid groups
    - The braid shelf
    - A freeness criterion
  - 3. The word problem of SD: a syntactic solution
    - The free monogenerated shelf
    - The comparison property
    - The Thompson's monoid of SD
- Minicourse II. Connection with set theory
  - 1. The set-theoretic shelf
    - Large cardinals and elementary embeddings
    - The iteration shelf
  - 2. Periods in Laver tables
    - Quotients of the iteration shelf
    - The dictionary
    - Results about periods

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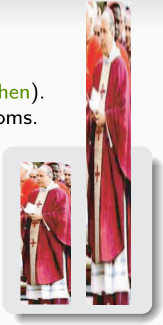

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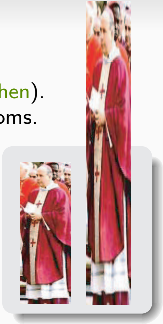

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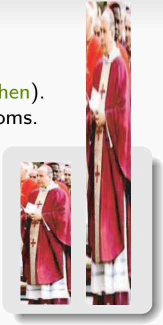

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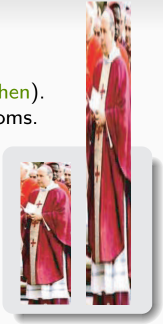

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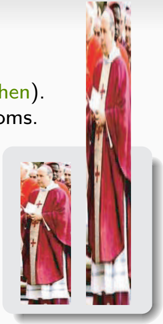

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  - A super-infinite set must be so large that it contains undefinable elements (since all definable elements must be fixed).

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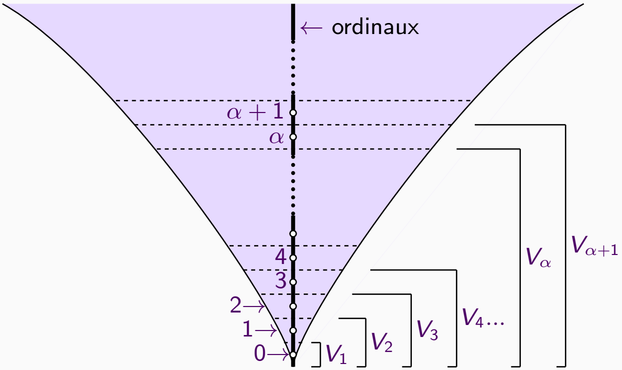

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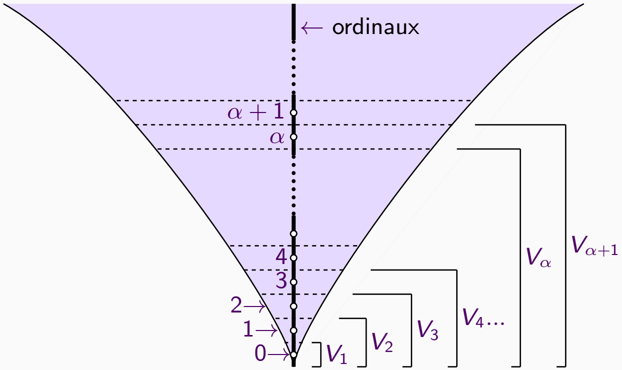

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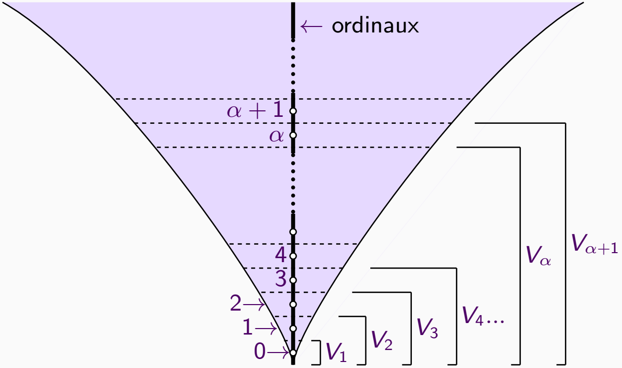

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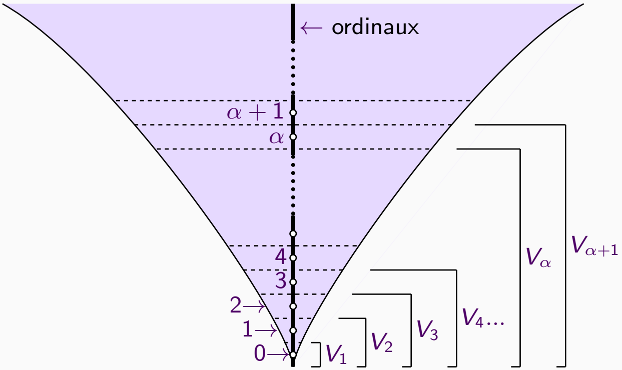

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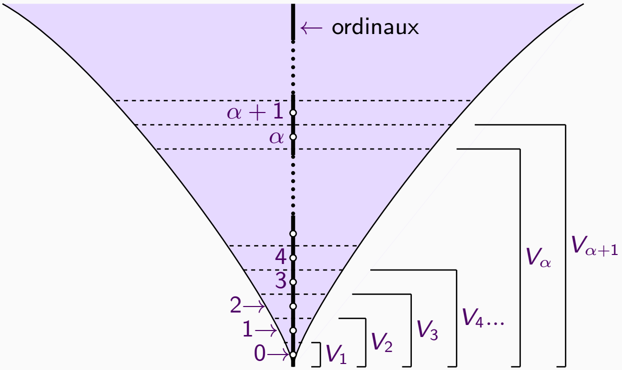

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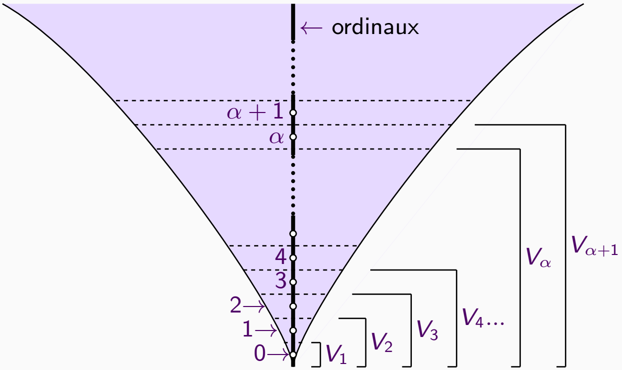

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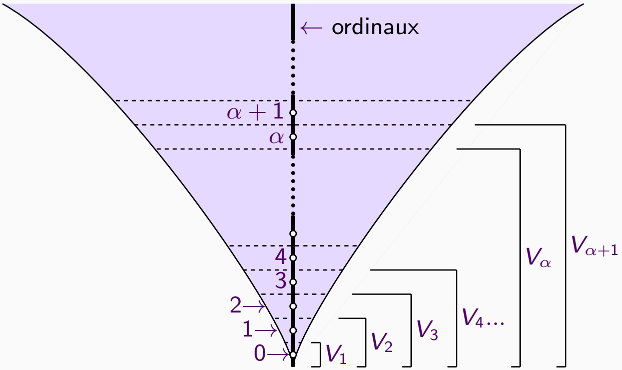

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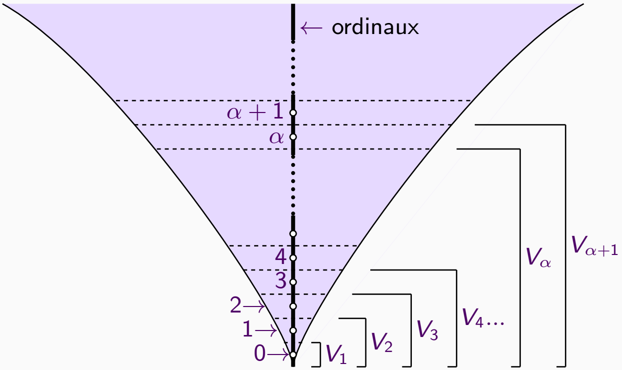

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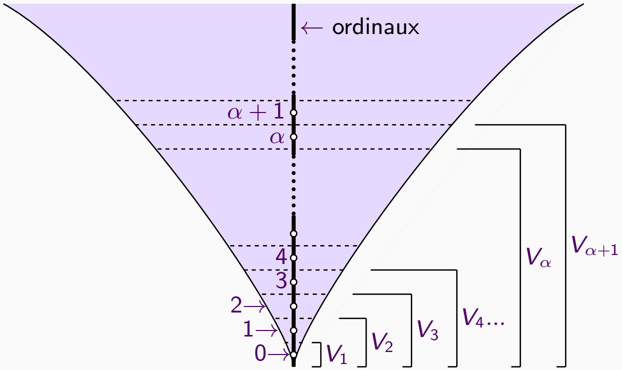

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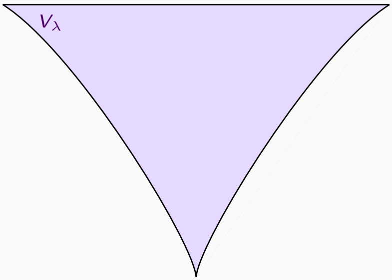

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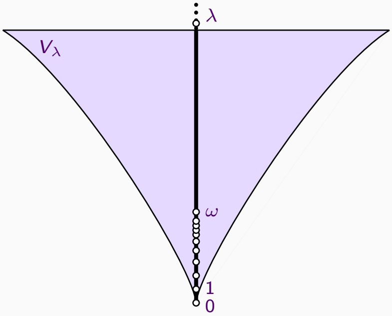

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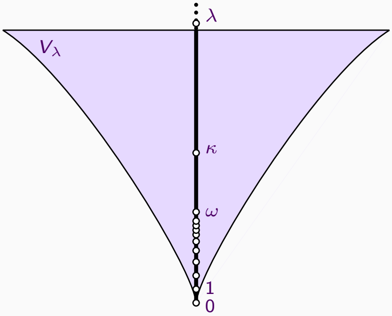

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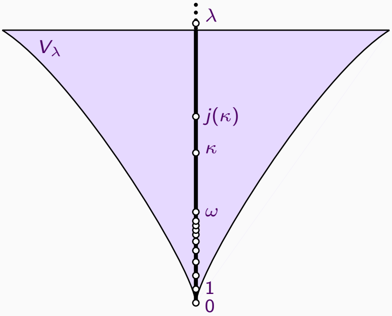

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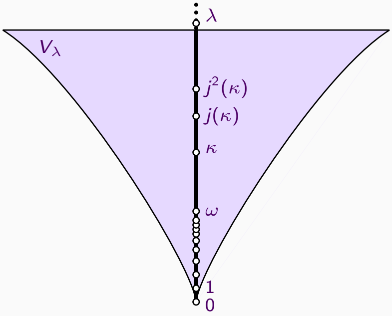

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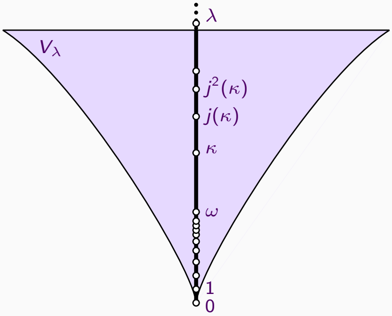

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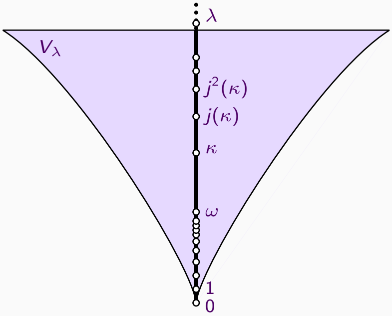

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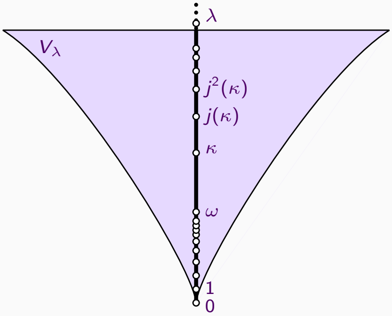

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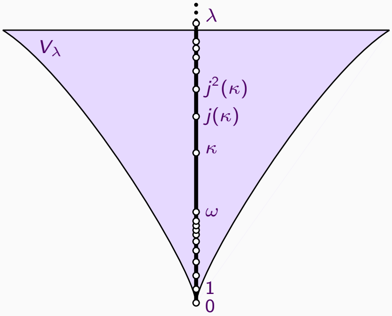

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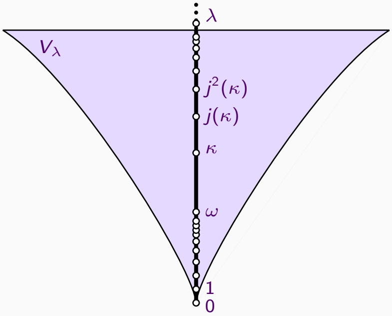

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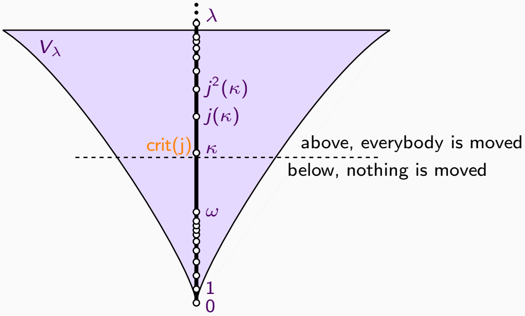

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  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall\alpha<\kappa\,(j(\alpha) = \alpha)$ implies $\forall\alpha<j(\kappa)\,(j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$.

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$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

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  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
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- Minicourse I. The SD-world
  - 1. A general introduction
    - Classical and exotic examples
    - Connection with topology: quandles, racks, and shelves
    - A chart of the SD-world
  - 2. The word problem of SD: a semantic solution
    - Braid groups
    - The braid shelf
    - A freeness criterion
  - 3. The word problem of SD: a syntactic solution
    - The free monogenerated shelf
    - The comparison property
    - The Thompson's monoid of SD
- Minicourse II. Connection with set theory
  - 1. The set-theoretic shelf
    - Large cardinals and elementary embeddings
    - The iteration shelf
  - 2. Periods in Laver tables
    - Quotients of the iteration shelf
    - The dictionary
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▶ Proof: (Difficult...) Starts from $j \equiv_{\mathrm{crit(i)}} i[j]$ and similar.
Uses in particular $\mathrm{crit(j_{[m]})} = \mathrm{crit_n(j)}$ with n maximal s.t. 2^n divides m. □

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  $t(1)^{A_n} = 2^n$ *is equivalent to* $\mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$; $(*)$

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_{\mathrm{n}}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_{\mathrm{n}}(\mathrm{j}). \qquad (**)$$

- Lemma: *For every* j *in* E_λ*, every term* $t(x)$*, and every* n*,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_\mathrm{n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$

- Lemma: *For every* j *in* E_λ, *every term* $t(x)$, *and every* n,

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_\mathrm{n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit}_\mathrm{n}(\mathrm{j})} \mathrm{id}$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$t(1)^{A_n} = 2^n$ *is equivalent to* $\mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}$; (∗)

$t(1)^{A_{n+1}} = 2^n$ *is equivalent to* $\mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}$. (∗∗)

  - Proof: For (∗): $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$,

- <u>Lemma</u>: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit_n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit_n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit_n}(\mathrm{j})} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit}_\mathrm{n}(\mathrm{j}). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit}_\mathrm{n}(\mathrm{j})}$ id,
  i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

  For $(**)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) = \mathrm{crit}_\mathrm{n}(\mathrm{j})$ is the conjunction
  of $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit}_\mathrm{n}(\mathrm{j})$ and $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \not\geqslant \mathrm{crit}_{\mathrm{n}+1}(\mathrm{j})$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) \geqslant \mathrm{crit_n}(\mathrm{j}); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter}(\mathrm{j})}) = \mathrm{crit_n}(\mathrm{j}). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit_n}(\mathrm{j})} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) = \mathrm{crit_n}(\mathrm{j})$ is the conjunction
of $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ and $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \not\geqslant \mathrm{crit_{n+1}}(\mathrm{j})$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$:

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t(j)}^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}}$ id,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction
of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) \geqslant \mathrm{crit_n(j)}; \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(\mathrm{t}(\mathrm{j})^{\mathrm{Iter(j)}}) = \mathrm{crit_n(j)}. \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ means $t(j) \equiv_{\mathrm{crit_n(j)}}$ id,
  i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

  For $(**)$: $\mathrm{crit(t(j))} = \mathrm{crit_n(j)}$ is the conjunction
  of $\mathrm{crit(t(j))} \geqslant \mathrm{crit_n(j)}$ and $\mathrm{crit(t(j))} \not\geqslant \mathrm{crit_{n+1}(j)}$, hence
  of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
  the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}

- Lemma: *For every* j *in* E_λ*, every term* $t(x)$*, and every* n*,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

  - Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
    i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

    For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction
    of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
    of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For* $m \leqslant n$ *and* $p \leqslant 2^n$*,*
  the period of p *jumps from* 2^m *to* 2^{m+1} *between* A_n *and* A_{n+1}
  iff $j_{[p]}$ *maps* $\mathrm{crit_m}(j)$ *to* $\mathrm{crit_n}(j)$*.*

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

  ▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
  i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.
  For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction
  of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
  of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
  the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
  iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.

  ▶ Proof: Apply the lemma to the term $x_{[p]}$.

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$, i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$, the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1} iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(j) = \mathrm{crit}(j_{[2^m]})$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \textit{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ means $t(j) \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(\mathrm{t}(\mathrm{j})) = \mathrm{crit_n}(\mathrm{j})$ is the conjunction
of $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \geqslant \mathrm{crit_n}(\mathrm{j})$ and $\mathrm{crit}(\mathrm{t}(\mathrm{j})) \not\geqslant \mathrm{crit_{n+1}}(\mathrm{j})$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit_m}(\mathrm{j})$ to $\mathrm{crit_n}(\mathrm{j})$.*

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(\mathrm{j}) = \mathrm{crit}(\mathrm{j}_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit_m}(\mathrm{j})$ to $\mathrm{crit}(\mathrm{j}_{[p]}[\mathrm{j}_{[2^m]}])$,
so the RHT is $\mathrm{crit}(\mathrm{j}_{[p]}[\mathrm{j}_{[2^m]}]) = \mathrm{crit_n}(\mathrm{j})$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit_n}(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also

$$(p \triangleright 2^m)^{A_{n+1}} = 2^n. \qquad (***).$$

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit}_n(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also

$$(p \triangleright 2^m)^{A_{n+1}} = 2^n. \qquad (***).$$

First, $(***)$ implies $\pi_{n+1}(p) > 2^m$.

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit_n}(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit_n}(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ means $t(j) \equiv_{\mathrm{crit_n}(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit_n}(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit_n}(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit_{n+1}}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit_n}(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit_m}(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit_m}(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit_n}(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also

$$(p \triangleright 2^m)^{A_{n+1}} = 2^n. \qquad (***).$$

First, $(***)$ implies $\pi_{n+1}(p) > 2^m$. Conversely, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$,

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit}_n(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also

$$(p \triangleright 2^m)^{A_{n+1}} = 2^n. \qquad (***).$$

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- Open questions: Find alternative proofs using no Laver cardinal.

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  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

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