The SD-world: a bridge between algebra, topology, and set theory

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- 1. Overview of the SD-world, with a special emphasis on the word probleme of SD.
- 2. The connection with set theory and the Laver tables.

Plan:

- Minicourse I. The SD-world
  - 1. A general introduction
    - Classical and exotic examples
    - Connection with topology: quandles, racks, and shelves
    - A chart of the SD-world
  - 2. The word problem of SD: a semantic solution
    - Braid groups
    - The braid shelf
    - A freeness criterion
  - 3. The word problem of SD: a syntactic solution
    - The free monogenerated shelf
    - The comparison property
    - The Thompson's monoid of SD
- Minicourse II. Connection with set theory
  - 1. The set-theoretic shelf
    - Large cardinals and elementary embeddings
    - The iteration shelf
  - 2. Periods in Laver tables
    - Quotients of the iteration shelf
    - The dictionary
    - Results about periods

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- The self-distributivity law SD:
  - left version: "left self-distributivity"

$$x(yz) = (xy)(xz) \qquad \text{(LD)}$$

or

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \qquad \text{(LD)}$$

  - right version: "right self-distributivity"

$$(xy)z = (xz)(yz) \qquad \text{(RD)}$$

or

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \qquad \text{(RD)}$$

- Definition: An LD-groupoid, or left shelf, is a structure $(S, \triangleright)$ with $\triangleright$ obeying (LD).
  An RD-groupoid, or shelf, is a structure $(S, \triangleleft)$ with $\triangleright$ obeying (RD).

- Definition: A rack is a shelf in which all right-translations are bijections.
  - Equivalently: $(S, \triangleleft, \overline{\triangleleft})$ with $\triangleleft$, $\overline{\triangleleft}$ obeying (RD) and, in addition

$$(x \triangleleft y) \,\overline{\triangleleft}\, y = x \quad \text{and} \quad (x \,\overline{\triangleleft}\, y) \triangleleft y = x.$$

- Definition: A quandle is an idempotent rack ($x \triangleleft x = x$ always holds).

- "Trivial" shelves: S a set, f a map $S \to S$, and $x \triangleleft y := f(x)$.
  - A rack iff f is a permutation of S.
  - In particular: the cyclic rack: $\mathbb{Z}/n\mathbb{Z}$ with $p \triangleleft q := p + 1$.
  - In particular: the augmentation rack: $\mathbb{Z}$ with $p \triangleleft q := p + 1$.

- Lattice shelves: $(L, \vee, 0)$ a (semi)-lattice, and $x \triangleleft y := x \vee y$.
  - Idempotent; never a rack for $\#L \geqslant 2$: always $0 \triangleleft x = x \triangleleft x$ $(= x)$.
  - A non-idempotent related example: B a Boolean algebra, and $x \triangleleft y := x \vee y^c$.
    (i.e., "$x \Leftarrow y$")

- Alexander shelves: R a ring, t in R, E an R-module, and $x \triangleleft y := tx + (1 - t)y$.
  - A rack (even a quandle) iff t is invertible in R.
  - In particular: symmetries in $\mathbb{R}^n$: $x \triangleleft y := -x + 2y$ ($\rightsquigarrow$ root systems).

- Conjugacy quandles: G a group, $x \triangleleft y := y^{-1}xy$.
  - Always a quandle.
  - In particular: the free quandle based on X when G is the free group based on X.
    ↑
    when viewed as $(Q, \triangleleft, \overline{\triangleleft})$: $(F_X, \triangleleft)$ is <u>not</u> a free idempotent shelf,
    it satisfies other laws: $x \triangleleft (y \triangleleft (y \triangleleft x)) = (x \triangleleft (x \triangleleft y)) \triangleleft (y \triangleleft x)$, ...
    (Drápal-Kepka-Musílek, Larue)
  - Variants: $x \triangleleft y := y^{-n}xy^n$, $x \triangleleft y := f(y^{-1}x)y$ with $f \in \mathrm{Aut}(G)$, ...

- Core (or sandwich) quandles: G a group, and $x \triangleleft y := yx^{-1}y$.

- Half-conjugacy racks: G a group, X a subset of G,
  and $(x, g) \triangleleft (y, h) := (x, h^{-1}y^{-1}gyh)$ on $X \times G$.
  - Not idempotent for $X \not\subseteq Z(G)$.
  - the free rack based on X when G is the free group based on X.

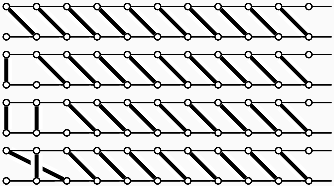
- The injection shelf: X an (infinite) set, $\mathfrak{I}_X$ monoid of all injections from X to itself,
  and $f \triangleleft g(x) := g(f(g^{-1}(x)))$ for $x \in \mathrm{Im}(g)$, and $f \triangleleft g(x) := x$ otherwise.
  - In particular, $X := \mathbb{N}$ $(= \mathbb{Z}_{>0})$ starting with sh : $n \mapsto n+1$:

sh :

sh $\triangleleft$ sh :

(sh $\triangleleft$ sh) $\triangleleft$ sh :

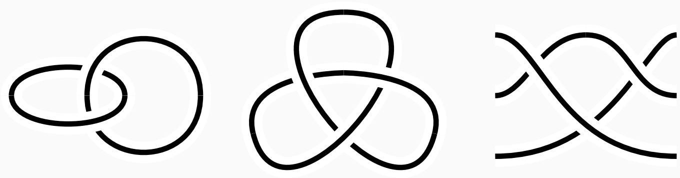
sh $\triangleleft$ (sh $\triangleleft$ sh) :



[P.D. *Algebraic properties of the shift mapping*, Proc. Amer. Math. Soc. 106 (1989) 617-623]

- The braid shelf, the iteration shelf, Laver tables: see below...

- Planar diagrams:



▶ projections of curves embedded in $\mathbb{R}^3$

- Generic question: recognizing whether two 2D-diagrams are (projections of) isotopic 3D-figures

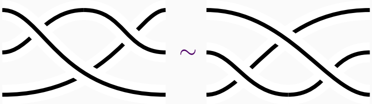
↑ continuously deform the 3D-figure allowing no curve crossing

▶ find isotopy invariants.

• Two diagrams represent isotopic figures iff one can go from the former to the latter using finitely many Reidemeister moves:

- type I : 

- type II : 

- type III : 

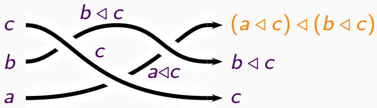
Connection with topology (3)

- Fix a set (of colors) S equipped with two operations $\lhd, \overline{\lhd}$,
  and color the strands in diagrams obeying the rules:

and

.

- Action of Reidemeister moves on colors:



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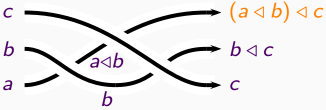
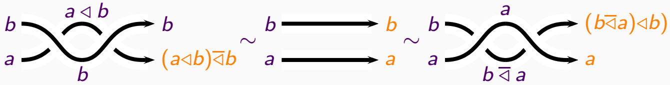


  - Hence:

> *$(S, \lhd)$-colorings are invariant under Reidemeister move III iff $(S, \lhd)$ is a shelf.*

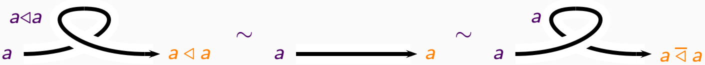
- Idem for Reidemeister move II:



  - Hence:

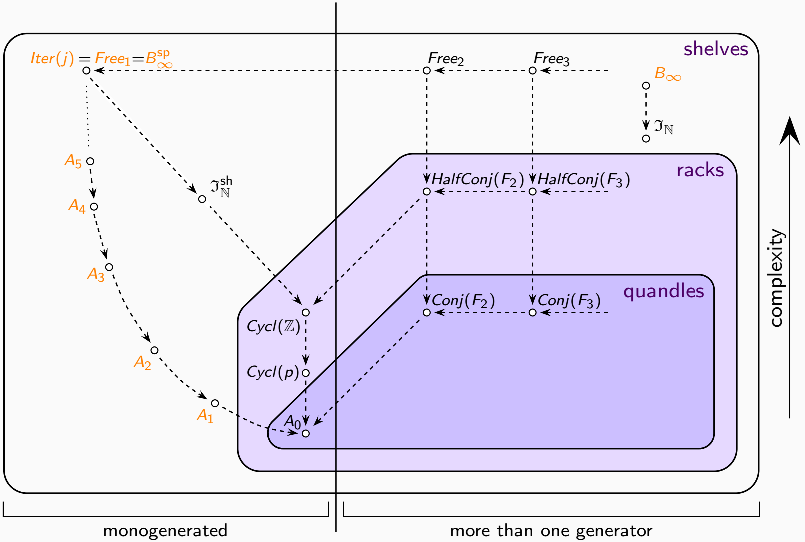
$(S, \triangleleft)$-colorings are invariant under Reidemeister moves II+III iff $(S, \triangleleft)$ is a rack.

- Idem for Reidemeister move I:



  - Hence:

$(S, \triangleleft)$-colorings are invariant under Reidemeister moves I+II+III iff $(S, \triangleleft)$ is a quandle.


shelves
racks
quandles
$Iter(j) = Free_1 = B_\infty^{sp}$
$Free_2$
$Free_3$
B_∞
$\mathfrak{I}_\mathbb{N}$
A_5
A_4
A_3
A_2
A_1
A_0
$\mathfrak{I}_\mathbb{N}^{sh}$
$HalfConj(F_2)$
$HalfConj(F_3)$
$Conj(F_2)$
$Conj(F_3)$
$Cycl(\mathbb{Z})$
$Cycl(p)$
complexity
monogenerated
more than one generator

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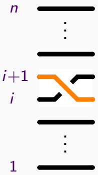
- Definition (Artin 1925/1948): The braid group B_n is the group with presentation

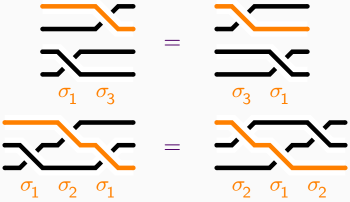
$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle.$$

$\simeq$ { braid diagrams } / isotopy:

$\sigma_i \quad \leftrightarrow$



- Example:

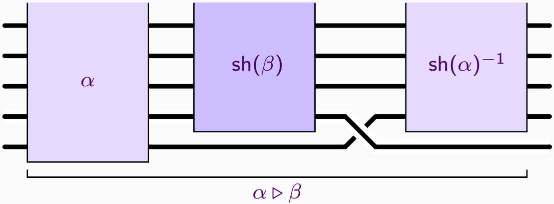


- Adding a strand on the right provides $i_{n,n+1} : B_n \hookrightarrow B_{n+1}$
  - Direct limit $B_\infty = \left\langle \sigma_1, \sigma_2, \ldots \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle$.
  - Shift endomorphism of B_∞: $\mathsf{sh} : \sigma_i \mapsto \sigma_{i+1}$.

- Proposition: For α, β in B_∞, define

$$\alpha \triangleright \beta := \alpha \cdot \mathsf{sh}(\beta) \cdot \sigma_1 \cdot \mathsf{sh}(\alpha)^{-1}.$$

Then $(B_\infty, \triangleright)$ is a left shelf.



- Examples: $1 \triangleright 1 = \sigma_1$, $\quad 1 \triangleright \sigma_1 = \sigma_2\sigma_1$, $\quad \sigma_1 \triangleright 1 = \sigma_1^2\sigma_2^{-1}$, $\quad \sigma_1 \triangleright \sigma_1 = \sigma_2\sigma_1$, etc.

▶ Proof: $\alpha \triangleright (\beta \triangleright \gamma) = \alpha \cdot \mathrm{sh}(\beta \cdot \mathrm{sh}(\gamma) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_2\sigma_1 \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$.

$(\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$
$= (\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\gamma)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}) \cdot\ \sigma_1 \cdot \mathrm{sh}(\alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1})^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}(\alpha)^{-1}\, \mathrm{sh}(\alpha)\, \mathrm{sh}^2(\gamma)\, \sigma_2\, \mathrm{sh}^2(\alpha)^{-1}\, \sigma_1\, \mathrm{sh}^2(\alpha)\, \sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha\, \mathrm{sh}(\beta)\, \sigma_1\, \mathrm{sh}^2(\gamma)\, \sigma_2\sigma_1\sigma_2^{-1}\, \mathrm{sh}^2(\beta)^{-1}\, \mathrm{sh}(\alpha)^{-1}$
$= \alpha \cdot \mathrm{sh}(\beta) \cdot \mathrm{sh}^2(\gamma) \cdot \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \cdot \mathrm{sh}^2(\beta)^{-1} \cdot \mathrm{sh}(\alpha)^{-1}$ □

- <u>Remark</u>: Shelf (=right shelf) with

$$\alpha \triangleleft \beta := \mathrm{sh}(\beta)^{-1} \cdot \sigma_1 \cdot \mathrm{sh}(\alpha) \cdot \beta,$$

but less convenient here.

- <u>Remark</u>: Works similarly with

$$x \triangleright y := x \cdot \phi(y) \cdot e \cdot \phi(x)^{-1}$$

whenever G is a group G, e belongs to G, and ϕ is an endomorphism ϕ satisfying

$$e\,\phi(e)\,e = \phi(e)\,e\,\phi(e) \quad \text{and} \quad \forall x\,(e\,\phi^2(x) = \phi^2(x)\,e).$$

• Proposition (D., 1989, Laver, 1989) *If $(S, \triangleright)$ is a monogenerated left shelf, a sufficient condition for $(S, \triangleright)$ to be free is that the relation $\sqsubset$ on S has no cycle.*

$\uparrow$

$x \sqsubset y$ if $\exists z\,(x \triangleright z = y)$.

- Equivalently: $x = (\cdots((x \triangleright z_1) \triangleright z_2) \triangleright \cdots) \triangleright z_n$ is impossible.

• Theorem (D., 1991): *Every braid in B_∞ generates in $(B_\infty, \triangleright)$ a free left shelf.*

- Typically: The subshelf of $(B_\infty, \triangleright)$ generated by 1 is a free left shelf.

- Proof (Larue, 1992): Use the (faithful) Artin representation ρ of B_∞ in $\mathrm{Aut}(F_\infty)$: $\rho(\sigma_i)(x_i) := x_i x_{i+1} x_i^{-1}$, $\rho(\sigma_i)(x_{i+1}) := x_i$, $\rho(\sigma_i)(x_k) := x_k$ for $k \neq i, i+1$,
Then $\alpha \sqsubset \beta$ in B_∞ implies that $\alpha^{-1}\beta$ has an expression with $\geqslant 1$ letter σ_1 and no σ_1^{-1}.
For such a braid γ, the word $\rho(\gamma)(x_1)$ in F_∞ finishes with the letter x_1^{-1}. □

• Corollary: (solution of the wp of SD) *Given two terms T, T':*

- *Evaluate T and T' at $x := 1$ in B_∞;*
- *Then $T =_{\mathrm{SD}} T'$ iff $T(1) = T'(1)$ in B_∞.*

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- Describe the free (left) shelf based on a set X ($=$ the most general shelf gen'd by X)
  ($=$ the shelf generated by X, every shelf generated by X is a quotient of)

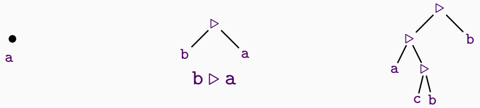
- Lemma: *Let $\mathcal{T}_X$ be the family of all terms built from X and $\triangleright$, and $=_{\mathrm{SD}}$ be the congruence* (i.e., compatible equiv. rel.) *on $\mathcal{T}_X$ generated by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $\mathcal{T}_X/{=_{\mathrm{SD}}}$ is the free left-shelf based on X.

  - ▶ Proof: trivial. □
  - ▶ ...but says nothing: $=_{\mathrm{SD}}$ not under control so far. In particular, is it decidable?

- Terms on X as binary trees with nodes $\triangleright$ and leaves in X: assuming $X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$,



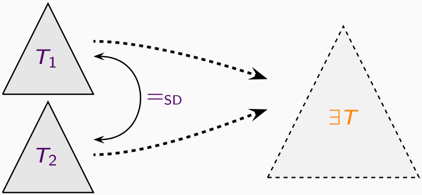
$\mathtt{b} \triangleright \mathtt{a}$ $\qquad$ $(\mathtt{a} \triangleright (\mathtt{b} \triangleright \mathtt{c})) \triangleright \mathtt{b}$

- Lemma (confluence): *Let* $\to_{\mathrm{SD}}$ *be the* *semi-congruence on* $\mathcal{T}_X$ *gen'd by all pairs*

$$(T_1 \triangleright (T_2 \triangleright T_3)\ ,\ (T_1 \triangleright T_2) \triangleright (T_1 \triangleright T_3)).$$

Then $T_1 =_{\mathrm{SD}} T_2$ *holds iff one has* $T_1 \to_{\mathrm{SD}} T$ *and* $T_2 \to_{\mathrm{SD}} T$ *for some* T*.*

"SD-equivalent iff admit a common SD-expansion"



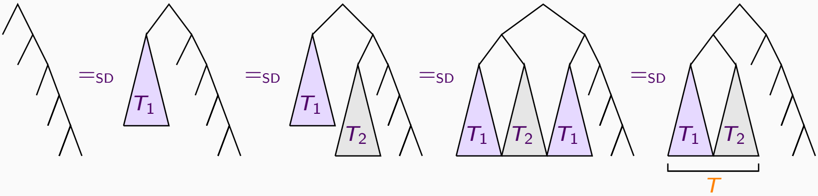
- Lemma (absorption): *Define* $x^{[1]} := x$ *and* $x^{[n]} := x \triangleright x^{[n-1]}$ *for* $n \geqslant 2$. *For* T *in* $\mathcal{T}_x$,

$$x^{[n+1]} =_{\mathrm{SD}} T \triangleright x^{[n]}$$

holds for $n > \mathrm{ht}(T)$, *where* $\mathrm{ht}(x) := 0$ *and* $\mathrm{ht}(T_1 \triangleright T_2) := \max(\mathrm{ht}(T_1), \mathrm{ht}(T_2)) + 1$.

▶ Proof: Induction on T. For $T = x$, direct from the definitions.
Assume $T = T_1 \triangleright T_2$ and $n > \mathrm{ht}(T)$. Then $n - 1 > \mathrm{ht}(T_1)$ and $n - 1 > \mathrm{ht}(T_2)$.

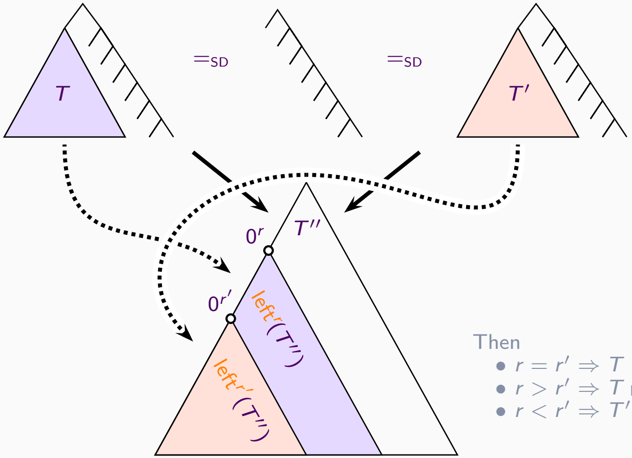
Then $x^{[n+1]} =_{\mathrm{SD}} T_1 \triangleright x^{[n]}$ by induction hypothesis for T_1
$=_{\mathrm{SD}} T_1 \triangleright (T_2 \triangleright x^{[n-1]})$ by induction hypothesis for T_2
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright (T_1 \triangleright x^{[n-1]})$ by applying SD
$=_{\mathrm{SD}} (T_1 \triangleright T_2) \triangleright x^{[n]}$ by induction hypothesis for T_1
$= T \triangleright x^{[n]}$. □



- **Lemma** (comparison I): *Write* $T \sqsubset_{SD} T'$ *for* $\exists T''\,(T' =_{SD} T \triangleright T'')$, *and* $\sqsubset^*_{SD}$ *for the transitive closure of* $\sqsubset_{SD}$. *Then, for all* T, T' *in* $\mathcal{T}_x$, *one has at least one of*

$$T \sqsubset^*_{SD} T', \quad T =_{SD} T', \quad T' \sqsubset^*_{SD} T.$$

  - Proof:



Then

- $r = r' \Rightarrow T =_{SD} T'$
- $r > r' \Rightarrow T \sqsubset^*_{SD} T'$
- $r < r' \Rightarrow T' \sqsubset^*_{SD} T$

□

• Application: If $(S, \triangleright)$ is a monogenerated left-shelf, any two distinct elements of S are $\sqsubset^*$-comparable (with $\sqsubset^*$= transitive closure of $\sqsubset$ = iterated left divisibility).

• Proposition (freeness criterion): *If $(S, \triangleright)$ is a monogenerated left-shelf and $\sqsubset$ has no cycle, then $(S, \triangleright)$ is free.*

▶ Proof: Assume S gen'd by g. "S is free" means "$T \neq_{\mathrm{SD}} T' \Rightarrow T(g) \neq T'(g)$". Now $T \neq_{\mathrm{SD}} T'$ implies $T \sqsubset^*_{\mathrm{SD}} T'$ or $T' \sqsubset^*_{\mathrm{SD}} T$,
whence $T(g) \sqsubset^* T'(g)$ or $T'(g) \sqsubset^* T(g)$.
As $\sqsubset$ has no cycle in S, both imply $T(g) \neq T'(g)$. □

• Proposition: *If there exists at least one shelf with $\sqsubset$ acyclic, then $\sqsubset^*_{\mathrm{SD}}$ has no cycle.*

▶ And such examples do exist: 1. Iteration shelf (Laver, 1989);
2. Free shelf (Dehornoy, 1991); 3. Braid shelf (D., 1991, Larue, 1992, D., 1994).

• Corollary: (solution of the wp of SD) *Given two terms T, T':*
- ▶ *Find a common LD-expansion T'' of $T \triangleright x^{[n]}$ and $T' \triangleright x^{[n]}$;*
- ▶ *Find r and r' satisfying $T \rightarrow_{\mathrm{SD}} \mathrm{left}^r(T'')$ and $T' \rightarrow_{\mathrm{SD}} \mathrm{left}^{r'}(T'')$.*
- ▶ *Then $T =_{\mathrm{SD}} T'$ iff $r = r'$.*

- Definition: For α a binary address (= finite sequence of 0s and 1s), let SD_α be the partial operator "apply SD in the expanding direction at address α". The Thompson's monoid of SD is the monoid $\mathcal{M}_{\mathrm{SD}}$ gen'd by all SD_α and their inverses.

- Fact: *Two terms T, T' are SD-equivalent iff some element of $\mathcal{M}_{\mathrm{SD}}$ maps T to T'.*

- Now, for every term T, select an element χ_T of $\mathcal{M}_{\mathrm{SD}}$ that maps $x^{[n+1]}$ to $T \triangleright x^{[n]}$.
  - Follow the inductive proof of the absorption property:

$$\chi_x := 1, \quad \chi_{T_1 \triangleright T_2} := \chi_{T_1} \cdot \mathrm{sh}_1(\chi_{T_2}) \cdot \mathrm{SD}_\emptyset \cdot \mathrm{sh}_1(\chi_{T_1})^{-1}. \qquad (*)$$

- Next, identify relations in $\mathcal{M}_{\mathrm{SD}}$:

$$\mathrm{SD}_{11\alpha}\mathrm{SD}_\alpha = \mathrm{SD}_\alpha\mathrm{SD}_{11\alpha}, \quad \mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_{0\alpha} = \mathrm{SD}_\alpha\mathrm{SD}_{1\alpha}\mathrm{SD}_\alpha, \text{ etc.} \qquad (**)$$

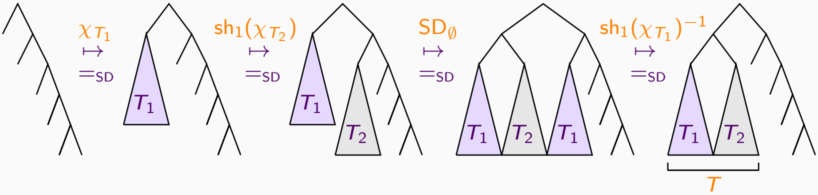
  - When every SD_α s.t. α contains 0 is collapsed, only the $\mathrm{SD}_{11...1}$s remain.
  - Write σ_{i+1} for the image of $\mathrm{SD}_{11...1}$, i times 1. Then $(**)$ becomes

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |j-i| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |j-i| = 1.$$

  - The resulting quotient of $\mathcal{M}_{\mathrm{SD}}$ is B_∞ (!).
  - If ϕ maps T to T', then $\mathrm{sh}_0(\phi)$ maps $T \triangleright x^{[n]}$ to $T' \triangleright x^{[n]}$, so collapsing all $\mathrm{sh}_0(\phi)$ must give an SD-operation on the quotient, i.e., on B_∞.
  - Its definition is the projection of $(*)$, i.e.,

$$a \triangleright b := a \cdot \mathrm{sh}(b) \cdot \sigma_i \cdot \mathrm{sh}(a)^{-1}.$$

- The "magic" braid operation revisited:

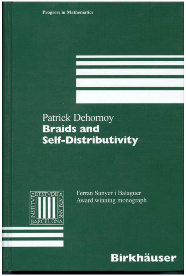


whence $\chi_{T_1 \triangleright T_2} = \chi_{T_1} \cdot sh_1(\chi_{T_2}) \cdot SD_\emptyset \cdot sh_1(\chi_{T_1}^{-1})$,

which projects to the braid operation.

.../...

- See more in [P.D., Braids and selfdistributivity, PM192, Birkhaüser (1999)]



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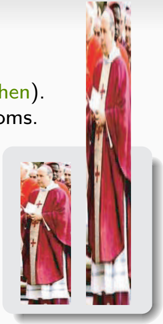


- Set theory is the theory of infinities.

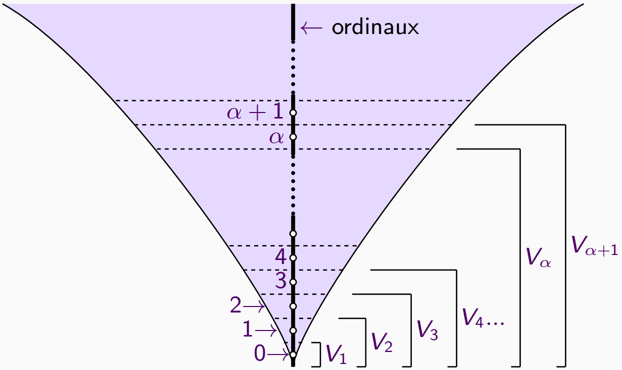
- The standard axiomatic system ZF is (very) incomplete (Gödel, Cohen).
  - Identify further properties of infinite sets = explore further axioms.
  - Typical example: axioms of large cardinal = solutions of
    $$\frac{\text{super-infinite}}{\text{infinite}} = \frac{\text{infinite}}{\text{finite}}$$
  - Set theory (as opposed to number theory) begins when "there exists an infinite set" is in the base axioms;
  - Repeat the process with "super-infinite".

- Principle: self-similar implies large
  - X infinite: $\exists j : X \to X$ (j injective not bijective)
  - X super-infinite: $\exists j : X \to X$ (j inject. not biject. preserving all $\in$-definable notions)

    ↑ an elementary embedding of X

- Example: $\mathbb{N}$ is not super-infinite.
  - A super-infinite set must be so large that it contains undefinable elements (since all definable elements must be fixed).

- Fact: *There is a canonical filtration of sets by the sets* V_α, α *an ordinal, def'd by*
  $V_0 := \emptyset$, $V_{\alpha+1} := \mathfrak{P}(V_\alpha)$, $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ *for* λ *limit.*



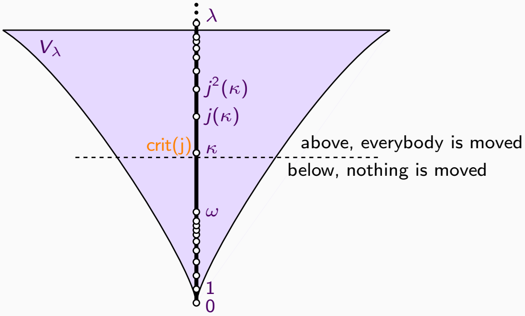
- Fact: *If* λ *is a limit ordinal and* $f : V_\lambda \to V_\lambda$,
  then $f = \bigcup_{\alpha<\lambda} f \cap V_\alpha^2$ *and* $f \cap V_\alpha^2$ *belongs to* V_λ *for every* $\alpha < \lambda$.

  ▶ Proof: Every element of V_λ belongs to some V_α with $\alpha < \lambda$; The set $f \cap V_\alpha^2$ is included in V_α^2, hence in $V_{\alpha+2}$, hence it belongs to $V_{\alpha+3}$, hence to V_λ. □

- Definition: *A Laver cardinal is a cardinal λ s.t. the set V_λ is "super-infinite", i.e., there exists a non-surjective elementary embedding from V_λ to itself.*

- Fact: *If there exists a super-infinite set, there exists a super-infinite set V_λ (hence a Laver cardinal).*

- Fact: *Assume $j : V_\lambda \to V_\lambda$ witnesses that λ is a Laver cardinal.*
  - *The map j sends every ordinal α to an ordinal $\geqslant \alpha$.*
  - *There exists an ordinal α satisfying $j(\alpha) > \alpha$.*
  - *There exists a smallest ordinal κ satisfying $j(\kappa) > \kappa$: the "critical ordinal" of j.*
  - *One necessarily has $\lambda = \sup_n j^n(\mathrm{crit(j)})$.*



- If λ is a Laver cardinal, let E_λ be the family of all non-trivial (= non-surjective) elementary embeddings from V_λ to itself (which is nonempty).

- Definition: For i, j in E_λ, the result of applying i to j is
$$i[j] := \bigcup_{\alpha<\lambda} i(j \cap V_\alpha^2).$$

- Lemma: *The map* $(i,j) \mapsto i[j]$ *is a binary operation on* E_λ*, and* $(E_\lambda, -[-])$ *is a left-shelf.*

  ▶ Proof: The sets $j \cap V_\alpha^2$ belong to V_λ, and are pairwise compatible partial maps, hence so are the sets $i(j \cap V_\alpha^2)$: so $i[j]$ is a map from V_λ to itself.
  "Being an elementary embedding" is definable, hence $i[j]$ is an elementary embedding.
  "Being the image of" is definable, hence $\ell = j[k]$ implies $i[\ell] = i[j][i[k]]$,
  i.e., $i[j[k]] = i[j][i[k]]$: the left SD law. □

- Attention! Application is not composition:
$$\mathrm{crit}(j \circ j) = \mathrm{crit}(j), \quad \text{but} \quad \mathrm{crit}(j[j]) > \mathrm{crit}(j).$$

  ▶ Proof: Let $\kappa := \mathrm{crit}(j)$. For $\alpha < \kappa$, $j(\alpha) = \alpha$, hence $j(j(\alpha)) = \alpha$, whereas $j(\kappa) > \kappa$, hence $j(j(\kappa)) > j(\kappa) > \kappa$. We deduce $\mathrm{crit}(j \circ j) = \kappa$.
  On the other hand, $\forall \alpha{<}\kappa\, (j(\alpha) = \alpha)$ implies $\forall \alpha{<}j(\kappa)\, (j[j](\alpha) = \alpha)$, whereas $j(\kappa) > \kappa$ implies $j[j](j(\kappa)) > j(\kappa)$. We deduce $\mathrm{crit}(j[j]) = j(\kappa) > \kappa$. □

- Proposition: If j is a nontrivial elementary embedding from V_λ to itself, then the iterates of j make a left-shelf $\mathrm{Iter}(j)$.

↑ closure of $\{j\}$ under the "application" operation: $j[j]$, $j[j][j]$...

- Theorem (Laver, 1989): *If j is a nontrivial elementary embedding from V_λ to itself, then $\sqsubset$ has no cycle in $\mathrm{Iter}(j)$; hence, $\mathrm{Iter}(j)$ is a free left-shelf.*

- A realization (the "set-theoretic realization") of the free (left)-shelf,
- ...plus a proof of that a shelf with acyclic $\sqsubset$ exists,
- ...whence a proof that $\sqsubset_{SD}$ is acyclic on $\mathcal{T}_x$,
- ...whence a solution for the word problem of SD (because both $=_{SD}$ and $\sqsubset^*_{SD}$ are semi-decidable).

but all this under the (unprovable) assumption that a Laver cardinal exists.

⇝ motivation for finding another proof/another realization...

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- Notation: ("left powers") $j_{[p]} := j[j][j]...[j]$, p times j.

- Definition: For j in E_λ,

  $\mathrm{crit_n(j)} :=$ the $(n+1)$st ordinal (from bottom) in $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$.

  ▶ One can show $\mathrm{crit_0(j)} = \mathrm{crit(j)}$, $\mathrm{crit_1(j)} = \mathrm{crit(j[j])}$, $\mathrm{crit_2(j)} = \mathrm{crit(j[j][j][j])}$, etc.

- Proposition (Laver, 1994): *Assume that λ is a Laver cardinal. Let j belong to E_λ. For i, i' in Iter(j) and $\gamma < \lambda$, declare $i \equiv_\gamma i'$ ("i and i' agree up to γ") if*

$$\forall x {\in} V_\gamma\, (i(x) {\cap} V_\gamma = i'(x) {\cap} V_\gamma).$$

Then $\equiv_{\mathrm{crit_n(j)}}$ is a congruence on Iter(j), it has 2^n classes, which are those of $j, j_{[2]}, ..., j_{[2^n]}$, the latter also being the class of id.

▶ Proof: (Difficult...) Starts from $j \equiv_{\mathrm{crit(i)}} i[j]$ and similar. Uses in particular $\mathrm{crit(j_{[m]})} = \mathrm{crit_n(j)}$ with n maximal s.t. 2^n divides m. □

- Recall: The Laver table A_n is the unique left-shelf on $\{1, ..., 2^n\}$ satisfying $p = 1_{[p]}$ for $p \leqslant 2^n$ and $2^n \triangleright 1 = 1$.

(or, equivalently, on $\{0, ..., 2^n{-}1\}$) satisfying $p = 1_{[p]}$ mod 2^n for $p \leqslant 2^n$ and $0 \triangleright 1 = 1$)

- Corollary: *The quotient-structure* $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ *is (isomorphic to) the table* A_n*.*
  - Proof: Write p for the $\equiv_{\mathrm{crit_n(j)}}$-class of $j_{[p]}$.
    The proposition says that $\mathrm{Iter(j)}/{\equiv_{\mathrm{crit_n(j)}}}$ is a left-shelf whose domain is $\{1, ..., 2^n\}$;
    By construction, $p = 1_{[p]}$ holds for $p \leqslant 2^n$.
    Then $j_{[2^n]} \equiv_{\mathrm{crit_n(j)}} \mathrm{id}$ implies $j_{[2^n+1]} \equiv_{\mathrm{crit_n(j)}} j$, whence $2^n \triangleright 1 = 1$ in the quotient. □

  - A (set-theoretic) realization of A_n as a quotient of the iteration shelf Iter(j).

- Lemma: *For every j in E_λ, every term $t(x)$, and every n,*

$$t(1)^{A_n} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) \geqslant \mathrm{crit}_n(j); \qquad (*)$$

$$t(1)^{A_{n+1}} = 2^n \quad \text{is equivalent to} \quad \mathrm{crit}(t(j)^{\mathrm{Iter}(j)}) = \mathrm{crit}_n(j). \qquad (**)$$

▶ Proof: For $(*)$: $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ means $t(j) \equiv_{\mathrm{crit}_n(j)} \mathrm{id}$,
i.e., the class of $t(j)$ in A_n, which is $t(1)^{A_n}$, is that of id, which is 2^n.

For $(**)$: $\mathrm{crit}(t(j)) = \mathrm{crit}_n(j)$ is the conjunction
of $\mathrm{crit}(t(j)) \geqslant \mathrm{crit}_n(j)$ and $\mathrm{crit}(t(j)) \not\geqslant \mathrm{crit}_{n+1}(j)$, hence
of $t(1)^{A_n} = 2^n$ and $t(1)^{A_{n+1}} \neq 2^{n+1}$: the only possibility is $t(1)^{A_{n+1}} = 2^n$. □

- Proposition ("dictionary"): *For $m \leqslant n$ and $p \leqslant 2^n$,*
the period of p jumps from 2^m to 2^{m+1} between A_n and A_{n+1}
iff $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}_n(j)$.

▶ Proof: Apply the lemma to the term $x_{[p]}$.
As $\mathrm{crit}_m(j) = \mathrm{crit}(j_{[2^m]})$, the embedding $j_{[p]}$ maps $\mathrm{crit}_m(j)$ to $\mathrm{crit}(j_{[p]}[j_{[2^m]}])$,
so the RHT is $\mathrm{crit}(j_{[p]}[j_{[2^m]}]) = \mathrm{crit}_n(j)$, whence $(1_{[p]} \triangleright 1_{[2^m]})^{A_{n+1}} = 2^n$ by $(**)$,
which is also

$$(p \triangleright 2^m)^{A_{n+1}} = 2^n. \qquad (***).$$

First, $(***)$ implies $\pi_{n+1}(p) > 2^m$. Conversely, $(***)$ projects to $(p \triangleright 2^m)^{A_n} = 2^n$,
implying $\pi_n(p) \leqslant 2^m$. As $\pi_{n+1}(p)$ is $\pi_n(p)$ or $2\pi_n(p)$, $(***)$ is equivalent to
the conjunction $\pi_n(p) = 2^m$ and $\pi_{n+1}(p) = 2^{m+1}$. □

- Lemma: *If j belongs to E_λ, for every $\alpha < \lambda$, one has*
$$j[j](\alpha) \leqslant j(\alpha).$$

▶ Proof: There exists β satisfying $j(\beta) > \alpha$, hence there is a smallest such β, which therefore satisfies $j(\beta) > \alpha$ and
$$\forall \gamma < \beta \; (j(\gamma) \leqslant \alpha). \qquad (*)$$
Applying j to $(*)$ gives
$$\forall \gamma < j(\beta) \; (j[j](\gamma) \leqslant j(\alpha)). \qquad (**)$$
Taking $\gamma := \alpha$ in $(**)$ yields $j[j](\alpha) \leqslant j(\alpha)$. □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(2) \geqslant \pi_n(1)$ holds for all n.*

▶ Proof: Write $\pi_n(1) = 2^{m+1}$, and let $\overline{n}$ be maximal $<n$ satisfying $\pi_{\overline{n}}(1) \leqslant 2^m$.
By construction, the period of 1 jumps from 2^m to 2^{m+1} between $A_{\overline{n}}$ and $A_{\overline{n}+1}$.
By the dictionary, j maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Hence, by the lemma, $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\leqslant \mathrm{crit}_{\overline{\mathrm{n}}}(\mathrm{j})$.
Therefore, there exists $n' \leqslant \overline{n} \leqslant n$ s.t. $j[j]$ maps $\mathrm{crit}_\mathrm{m}(\mathrm{j})$ to $\mathrm{crit}_{n'}(\mathrm{j})$.
By the dictionary, the period of 2 jumps from 2^m to 2^{m+1} between $A_{n'}$ and $A_{n'+1}$.
Hence, the period of 2 in A_n is at least 2^{m+1}. □

- Lemma: *If j belongs to E_λ, then λ is the supremum of the ordinals* $\mathrm{crit_n(j)}$.
  - Not obvious: $\{\mathrm{crit(i)} \mid \mathrm{i} \in \mathrm{Iter(j)}\}$ is countable, but its order type might be $>\omega$.
  - Proof: (difficult...) □

- Proposition (Laver): *If there exists a Laver cardinal, $\pi_n(1)$ tends to ∞ with n.*
  - Proof: Assume $\pi_n(1) = 2^m$. We wish to show that there exists $\overline{n} \geqslant n$ s.t. $\pi_{\overline{n}}(1) = 2^m$ and $\pi_{\overline{n}+1}(1) = 2^{m+1}$.

  By the dictionary, this is equivalent to j mapping $\mathrm{crit_m(j)}$ to $\mathrm{crit_{\overline{n}}(j)}$.
  Now j maps $\mathrm{crit_m(j)}$, which is $\mathrm{crit(j_{[2^m]})}$, to $\mathrm{crit(j[j_{[2^m]}]}$.
  As $j[j_{[2^m]}]$ belongs to $\mathrm{Iter(j)}$, the lemma implies $\mathrm{crit(j[j_{[2^m]}]} = \mathrm{crit_{\overline{n}}(j)}$ for some $\overline{n}$. □

- Open questions: Find alternative proofs using no Laver cardinal.

- Are the properties of Laver tables an application of set theory?
  - So far, yes;
  - In the future, formally no if one finds alternative proofs using no large cardinal.
  - But, in any case, it is set theory that made the properties first accessible.

- Even if one does not believe that large cardinals exist (or are interesting), one should agree that they can provide useful intuitions.

- An analogy:
  - In physics: using a physical intuition, guess statements, then pass them to the mathematician for a formal proof.
  - Here: using a logical intuition (existence of a Laver cardinal), guess statements (periods tend to ∞ in Laver tables), then pass them to the mathematician for a formal proof.

- The two main open questions about Laver tables:
  - Can one find alternative proofs using no large cardinal? (as done for the free shelf using the braid realization)
  - Can one use them in low-dimensional topology?

Richard Laver (1942-2012)